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ABSTRACT

This book on propositional reasoning is part of a series designed to assist beginning and experienced teachers in becoming regular evaluators of their own instruction. The book consists of two parts. The first part provides information on probabilistic, disjunctive, combinatorial, implicational, deductive, and proportional reasoning and on controlling and manipulating variables. The second and longer part consists of 10 self-directed and self-instructive workshops. The goals of these workshops are to: (1) use an assessment method to administer, at any grade level, diagnostic tasks related to propositional reasoning; (2) identify the developmental level attained by each student in terms of propositional reasoning abilities; and (3) select, from resources available, subject matter (content and manipulative materials) appropriate to the identified levels of development. Except for the first workshop, the order in which workshops are completed or the number completed are at the discretion of the teacher. A short review of some research on propositional abilities and a bibliography are included. Also included (in an appendix) are examples of various types of problems related to propositional abilities. (JN)

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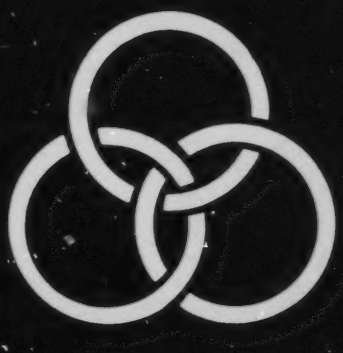

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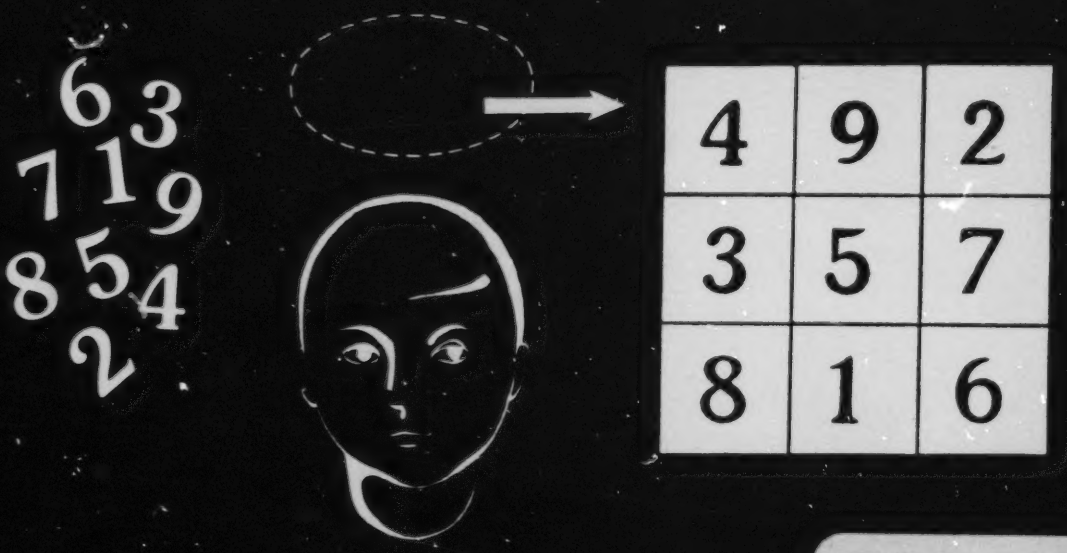
Learning about learning:

Propositional abilities



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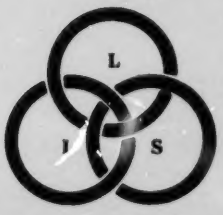

The mission of the Professional Development and Applied Research Center (PDARC) is to explore the components which contribute to learning at all levels—early childhood to adult. There are three components: The **Learner** (L), **Instruction** (I), and **Subject Areas** (S). Research and application in these interrelated areas can promote, for the teacher, a comprehensive approach to daily classroom situations and educational problems. The interwoven rings link the relationships of the themes in such a way that if any one ring is broken, the entire concept falls apart.

learning about learning

propositional reasoning

A PERSONAL WORKSHOP

by

Lawrence F. Lowery

This personal workshop focuses upon the identification of a set of abilities related to learning. The set, described as propositional reasoning, is not independent of but is distinct from other sets of abilities described in other books in the "*Learning About...Series.*" Although you will learn many ideas and develop various skills from experiencing this workshop, three major goals are identified below.

As a result of this workshop, you will be able to:

1. use an assessment method to administer, at any grade level, diagnostic tasks related to propositional reasoning.
2. identify the developmental level each student in your class has attained in propositional reasoning.
3. select, from resources available, subject matter (content and manipulative materials) appropriate to identified levels of development.

Hopefully you will become able to determine levels of development by observing students during their normal activities without relying upon the administration of the tasks.

This workshop is self-directed and self-instructive. It utilizes written materials, manipulative materials, videotapes, and students in your classroom. It suggests ideas for teaching experiences and discussions with colleagues. The value of this and other books in the series depends upon the use of the presented ideas in instructional situations.

This book on Propositional Abilities, developed by the Educational Research and Applications Program, is one of several under the general heading "*Learning About...Series.*" The major objective of this series is to assist beginning and experienced teachers in becoming regular evaluators of their own instruction. Users of this series are taught to use materials and techniques that help assess stages of intellectual development, and to use techniques for assessing their interactions with students.

Many individuals contributed to the content and the field testing of this book. Others checked for detail and accuracy. Without their assistance and expertise, this book could not have been developed.

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INTRODUCTION

Careful researchers and general observers have well substantiated the idea that fundamental aspects of human learning progress along a continuum that gradually enables individuals to cope more effectively and flexibly with situations and events within their environments. Along this continuum there are levels or way stations in which particular sets of abilities and behaviors are so well defined that each is referred to as a stage of development. The progression along this continuum and through the identified patterns of stages is a universal and invariant process that can be observed in natural, untutored settings. The patterns are the same for all individuals whether they are raised within a primitive culture or a culture of advanced technology.

The biological basis underlying stages along the continuum is established by periodic increases in brain size (Epstein 1974), brain weight (Epstein 1974), cellular growth within the brain (Winick and Rosso 1969), electrical functioning within the brain (Monnier 1960), and head circumference (Eichorn and Bayley 1962). The psychological basis for the stages is established through evidence of the individual's capacity to deal with independent ideas and to relate the ideas in increasing combinations in two-year spurts from about age three through 17 (Pascual-Leone 1970, Case 1974) and the individual's tendency to give the same kinds of incorrect views of the world as other individuals within two- to three-year ranges and as they grow older replace each view by a more sophisticated view which, in turn, is replaced (Piaget 1969). Although different researchers have provided various descriptions of the stage phenomenon (Bruner 1964, Erikson 1950, Gagne 1970, Vygotsky 1974) the person most noted for carefully describing the sequence in terms of periods and stages is Jean Piaget, the Swiss psychologist. His work set a descriptive foundation and vocabulary that enables educators and psychologists to identify, discuss, explore, and apply information about the stages.

It is not the purpose of this book to present the theoretical ideas of Piaget. These ideas are spread over more than 30 volumes and many articles. It is hoped that the reader will turn to some of the original works to better appreciate and evaluate the spirit of Piaget's enterprise, the nature of his evidence, and the validity of his conclusions.

The Concept of Stages

The term *stage* has been used in the literature without much consistency in definition. In this and other books in the "*Learning about...Series,*" the term is reserved for qualitatively defined strands of abilities developing through progressive interrelationships. One strand deals with classification abilities. Others deal with seriation abilities, conservation abilities, and propositional reasoning. Conceptual links along any one of these strands allow each stage to be described as a set of behaviors sharing some common feature. As such, a stage is a construct within a structurally defined system upon which there is a unification of a specific set of abilities.

Sequential progression alone is not a sufficient definition of stages. A definition must include the patterning of responses throughout the sequence. The findings of Shirley (1931) represent an invariant sequence—the behaviors characteristic in the development of locomotion in infancy. However, the components of the sequence are essentially a set of discrete points which bear no clear structural relationship from one to another. Sitting up, for example, does not necessarily have a relationship to holding a hand. Although the sequence is invariant, thus predictable, its components are not stages since there is no structural relationship among them. A definition must include the patterning of responses throughout the sequence.

Stages in this book involve sequentiality, a hierarchical integration of lower into higher levels, gradual consolidation in formation, and a structural whole character (unifying interrelated behaviors, concepts, or skills) that results in a broad structural network of interrelated abilities appearing, not all at once, but within a narrowly delimited period with further progress along any strand being deferred until the consolidation of this network.

> *Although at any stage of development, nature is preparing us for the next stage, the beauty of the system is that we are conscious of none of this. Ideally, we must fully accept and exist within our developmental stage and respond fully to its content and possibilities. This means that every stage is complete and perfect within itself. The three-year-old is not an incomplete five-year-old; the child is not an incomplete adult. Never are we simply on our way; always we have arrived. Everything is preparatory to something else that is in formation.*
>
> Pierce 1977

Stages in the Propositional Reasoning Strand

The term *propositional* is used to describe the integration of a particular set of cognitive abilities and operations. These abilities and operations are revealed when a student can think systematically in pure abstract terms. Such a student has developed the capacity to deal verbally with symbols and ideas without the necessity for an intervening manipulation of physical objects or a recent experience with them (reality becomes secondary to possibility). The individual can see given facts (reality) as the actually attained part of a complete set of possible transformations and can find the entire set of these transformations by hypothesizing. For example, if a student attempts to predict which objects in a collection of objects will float and which will sink, he or she can generate a set of possibilities or hypotheses such as "*sinking is caused by the object's weight, or by its size, or by its weight in relation to its size, or by some other variables,*" then develop a plan for systematically testing the hypotheses in ways that will confirm or disprove each conclusively. This capacity to reason about possibilities or hypotheses, and not just about specific objects at hand, is the most distinctive feature characterizing propositional reasoning.

Another important characteristic of propositional reasoning is the understanding that cognitive operations are performed on other cognitive operations, as opposed to concrete operations that are performed directly on objects and events. For example, if a student is to understand the formula for specific gravity,

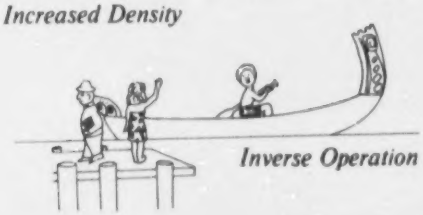
$$specific\ gravity = \frac{density\ of\ a\ substance}{density\ of\ water}$$

the student must first understand the formula for density.

$$density = \frac{weight}{volume}$$

To understand the formula for density, the student must utilize two reversible operations.

Increased Density



Inverse Operation

Reciprocity Operation

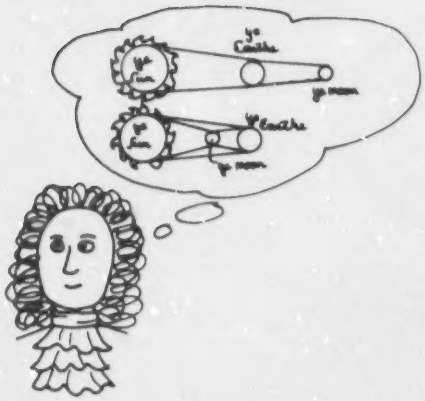
Combination of Both

If, according to the formula, the density of an object can be increased by adding weight while holding volume constant, it follows that this increase in density can be nullified either by removal of weight (inverse operation) or by a compensating increase in volume (reciprocity operation) or by certain combinations of both. Thus, understanding the specific gravity formula requires the ability to relate the two densities, or the use of operations upon operations.

Propositional reasoning and the operations performed with them generally emerge from age 12 to 18 and constitute the heart of Jean Piaget's Formal Operational Stage of cognitive development. Piaget separates this stage into two substages, which he labels IIIa and IIIb.

The first substage (IIIa) is distinguished by the appearance of a kind of reasoning based upon the logic of all possible combinations. For example, whether experimental or verbal problems are under consideration, combinatorial analysis (regard for all possible combinations) becomes an integral part of their solutions. The student can be called upon to manipulate physically all combinations, as in the chemical substances task of Inhelder and Piaget (see Combinatorial Reasoning Task in this book), or the student can be asked to identify mentally all combinations from verbal problems, such as those developed by Morf (see Implicational Reasoning Task). The basic power of propositional reasoning rests in being able to generate a set of hypotheses that are compatible with what is already known about the problem and the implications of which can be tested by experiment. By testing the implications of the hypotheses, new information is learned about the problem.

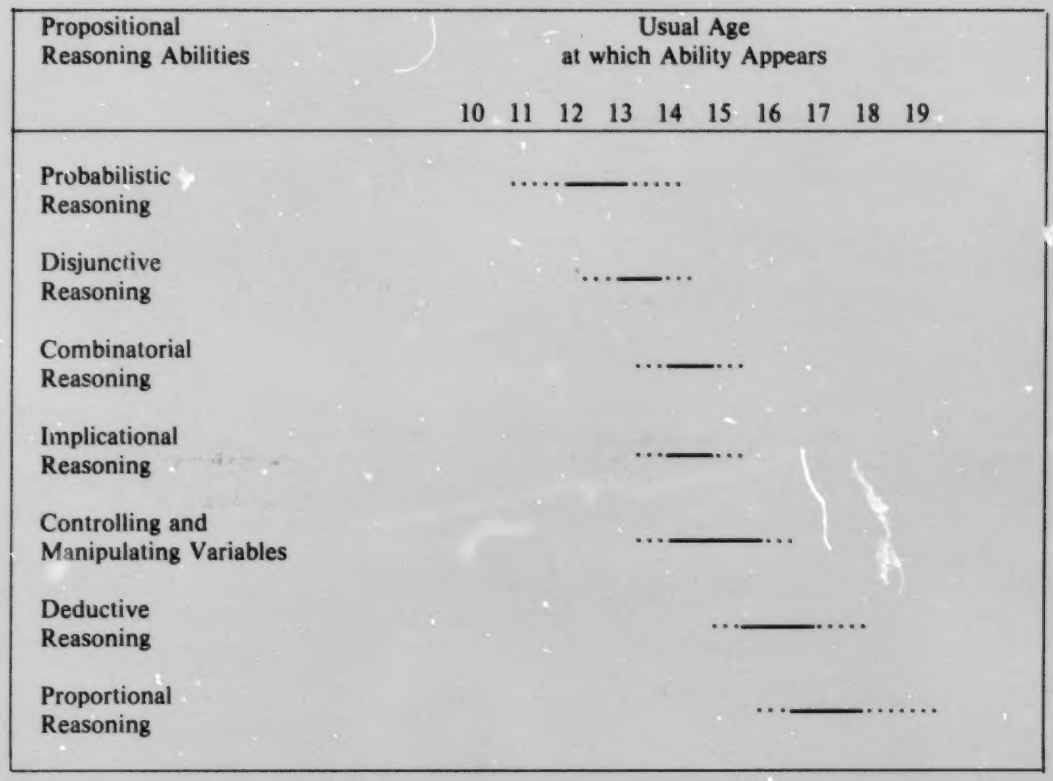
Making its appearance during the transition from substage IIIa to substage IIIb is the ability to separate variables by exclusion (the systematic testing of each variable individually while all others are held constant in order to determine which are relevant and which have no effect). Instead of being closely bound to immediate experience, the student can suggest, "*Maybe if I changed this, such and such would happen,*" and then perform the necessary actions to confirm or disprove the supposition. Inhelder and Piaget's pendulum problem is an example of this exclusion principle (see Controlling and Manipulating Variables Task).



The second substage (IIIb) is distinguished by the development of an individual's ability to organize reasoning along the direction of particular logical operations, such as conjunction ("*It must be A and B.*"), disjunction ("*It must be either A or B.*"), negation ("*It is neither A nor B.*"), and implication ("*If it is A, then B will be true.*"). As the student learns to think in increasingly abstract terms, the ability to combine such cognitive operations simultaneously is attained.

Attainment of Piaget's final stage does not mean that intellectual development is at its peak. The mind continues to develop beyond adolescence. However, later changes in cognition are simply refinements of the kind of thinking which becomes more or less established by about age 15. There are no more major qualitative changes in the nature of cognition such as those which occur during childhood.

Probable Sequence in Which Propositional Reasoning Abilities Appear



Summary

With the attainment of propositional abilities, the student can utilize certain basic strategies of scientific thinking, such as the scheme of "*all other things being equal.*" The student is able to generate hypothetical possibilities, to reason about them with completely coordinated reversibility, and to perform operations on other operations.

propositional tasks

With a friend, take turns trying each of the propositional ability tasks in this section.

At first, try to use the suggested dialogue that accompanies each task.
Then gradually incorporate your own style.

Probabilistic Reasoning

Principle

Probability is the relative frequency with which an event occurs or is likely to occur. Probabilistic reasoning involves mentally separating what is possible, what is real, and what is deductively necessary. At the Formal Operational Stage, this begins with the identification of possible events and ends with reality conceived as a realized part of the total number of possible events. For example, a student with this ability, given data that the number *three* is one of six numbers on a die which are equally likely to occur when the die is rolled, will predict in advance of rolling that the number *three* will appear on the average once in every six or ten in every sixty rolls of the die. Most students attain this ability by age 12 - 13.

Materials

2 opaque medium-sized paper bags.
12 blue (or other color) poker chips.
12 white (or other color) poker chips.
(Other objects can be substituted.)

Task*

The student (S) is sitting opposite the experimenter (E). E places the blue and white chips in two separate piles of twelve chips each, off to the side, but within reach of E and S.

Procedure

This task includes four separate parts. After each part, E may elect to discontinue the experiences depending on the performance of S. Guidelines for continuing or discontinuing are provided at the end of each part.

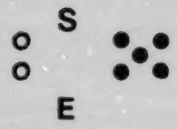
Part I

E arranges two separate groups of poker chips between E and S, one containing five white chips, the other two blue chips. E begins: "*Here is a set of five white poker chips and two blue poker chips.*"

1. E places all seven chips in a paper bag and shakes them up. As this is done, E states: "*I am putting the chips in this bag and shaking it. Now, if I were to take out one chip without first looking at its color, would I be more likely to get a white chip or a blue chip?*" E allows S to respond, then asks, "*How do you know?*" E again allows S to respond, then returns all chips to the piles.

*This task was adapted by permission from the research of Read Tuddenham, Professor of Psychology, University of California, Berkeley, California.

2. E now constructs two separate groups of chips, one containing two white chips, the other five blue chips, between E and S. E says: "*Now I have two white chips and five blue ones. If I put these chips in the bag as before and shake them up, would I be more likely to get a blue chip or a white?*" E allows S to respond, then explores S's reasoning.



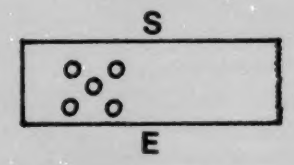
Guidelines for Proceeding

An acceptable response by S in either of the above experiences would be that the likelihood of drawing out a chip of a given color is greater if the bag contains more chips of that color than of the other. If S does not respond in an acceptable manner in either experience, E discontinues the task. Otherwise, E continues with Part II.

Part II

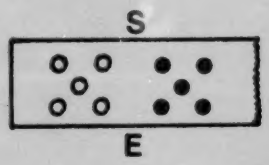
1. E places five white chips in a bag and sets the bag in front of S.

   E: "*There are five white chips in the bag. I want you to add blue chips to the bag so that if we shook them in the bag and drew one out, we would be* just as likely *to get a blue chip as a white.*" E allows S to respond, then asks: "*Why did you do what you did?*"



2. E next adjusts the number of chips in the bag so there are five blue and five white chips.

   E continues: "*There are now five white chips and five blue chips in the bag. Can you change the blue chips so that you are* more likely *to get a white than a blue,* without *changing the whites you have?*" E allows S to respond, then asks: "*Why did you do what you did?*"



Guidelines for Proceeding

S passes the first experience if he adds five blue chips to the bag and states that because there are now five chips of each color, the likelihood of drawing either color is the same. S passes the second experience if he removes any number of blue chips from the bag and states that because there are now more white chips than blue chips, the likelihood of drawing a white chip is now greater. If S passes at least one of the experiences, E continues with Part III. If S does not pass, discontinue the task.

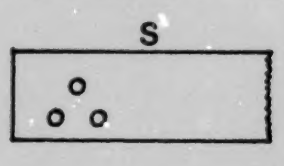
Part III

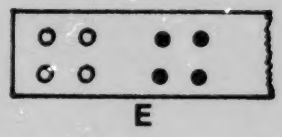
E places one empty bag before S and one before E.

E: "*Here is a bag for you and a bag for me. We will each have a set of chips* on *our bags. We will change our sets in different ways, but we will make it a rule that we always have at least one blue chip and one white chip in each set. Do you understand?*"

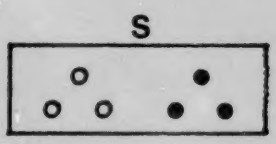
E allows S to respond, then places two separate sets of chips, one containing four whites, the other four blues, on E's bag. E then places one set of three white chips on S's bag.

1. E: "*This time I have made a set of four whites and four blues on my bag. You are to start with three whites on your bag. Don't change your whites. Now, using blue chips from the pile of extras, make a set of blue chips so that you are* just as likely *to get a blue as I am.*" E allows S to respond, then asks: "*Why did you do what you did?*"

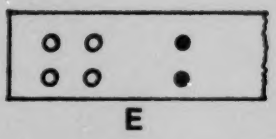




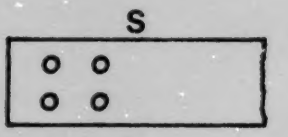
After S responds, E adjusts S's chips, if necessary, so that S has two separate sets of three chips each, one white, one blue. E then adjusts E's chips so that E has one set of four white chips and one set of two blue chips.



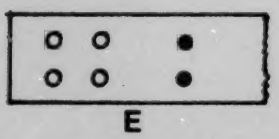
2. E continues: "*I have changed my set to have four whites and two blues. Your set has three whites and three blues. Now, change your set by adding whites so that you are* just as likely *to get a white as I am.*" E allows S to respond, then asks: "*Why did you do what you did?*"



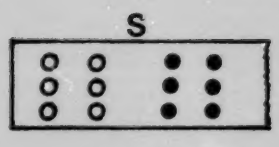
E allows S to respond, then proceeds, retaining E's one set of four white chips and one set of two blue chips. E adjusts S's chips so that S has one set of four white chips.



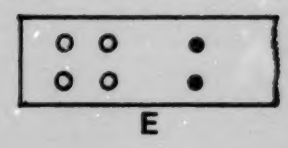
3. E continues: "*Now you have only four white chips. Add as few blue chips as needed to give yourself a greater chance than I have of drawing a blue chip. Don't change the whites.*" E allows S to respond, then asks: "*Why did you do what you did?*"



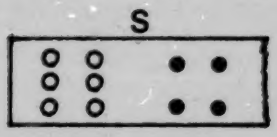
E allows S to respond. E retains one set of four white chips and another set of two blue chips, but now adjusts S's chips so that S has one set of six white chips and another set of six blue chips.



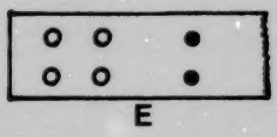
4. When this is done, E continues: "*I have left my set the same. Your set now has six whites and six blues. Change it, without changing the whites, so that you are* just as likely *as I am to draw a blue.*" E allows S to respond, then asks: "*Explain why you did what you did.*"



E allows S to respond. E retains E's chips as before, but adjusts S's chips so that S has one set of six white chips and one set of four blue chips.



5. E continues: "*This time I want you to change your set so that you are* more likely *than I am to draw a white. You can change either whites or blues but not both. Don't change any more chips than you have to, and remember the rule that there must be at least one chip of each color.*" E allows S to respond, then asks: "*Why did you do what you did?*"



Guidelines for Proceeding

S passes the first experience if he makes a separate set of three blue chips and explains that since each of S's sets now contain an equal number of chips, the likelihood of drawing from one set is equal to that of drawing from the other.

S's response in the second experience is passing if S adds three whites to S's set of whites and explains that since the number of whites is now double the number of blues in both S's and E's sets of chips, the likelihood of drawing whites is twice that of drawing blues for both S and E.

S's response to the third experience is passing if S makes a set of three blue chips and explains that since the number of blues S now has is more than half of the number of whites S has, the likelihood of S drawing a blue is more than it is for E (The number of blues E has is still exactly one-half the number of whites E has).

S's response to the fourth experience is passing if S removes three blue chips from S's set and explains that since the number of blues S now has is exactly one-half the number of whites S has, the likelihood of drawing a blue is the same as it is for E (E's blues still number exactly one-half that of E's whites).

S's response to the fifth experience is passing if S changes S's chips so that the number of whites is at least one more than twice the number of blues when S has completed the task and explains that since the whites now outnumber the blues by more than two-to-one, S is more likely to draw a white than E (E's whites still outnumber E's blues by exactly two-to-one). If S responds incorrectly to this last experience, E discontinues the tasks. Otherwise, E leaves E's and S's chips unchanged and continues with Part IV.

Part IV

E: "*You did the last experience by changing whites (or blues). Now I would like you to do it by changing the other color. Remember, you are to be more likely than I am to draw a white, and you are to change as few chips as necessary to do this.*" E allows S to respond, then asks: "*Explain why you did what you did.*"

Guidelines for Proceeding

A passing response is the same as for the last experience in Part III, except that S uses a different color.

Analysis

If S is at the Probabilistic Reasoning Stage of development, S will estimate probability (or likelihood) as a relationship between reality (what is given) and what is possible relative to the task presented. Consequently, S can predict the likelihood that an event will occur at some frequency relative to the occurrence of another event by comparing *real instances* of each, and then projecting or predicting their relative likelihood of occurrence in other possible situations. For example, a student at the Probabilistic Reasoning Stage, given a real sample of five blue poker chips and ten white chips from a universe of 100 mixed white and blue poker chips, will predict that if 60 poker chips are drawn (a possible situation), then on the average, twenty will be blue while forty will be white (an estimate of probability or likelihood).

If S is at the pre-Probabilistic Reasoning Stage of development, S will not be able to estimate probability in the absence of a concrete model. For example, a student at this level, given a real sample of five blue and ten white poker chips from the same universe of 100 mixed blue and white poker chips mentioned above, might require not only the information that 60 chips will be drawn, but that the 60 chips be drawn in fact, and the blue chips counted before the student will be able to determine the number of white chips in the sample.

In essence, the pre-stage student will revert to a simple additive model, rather than using a combinatorial one.

Disjunctive Reasoning

Principle

Solving problems based on factors that are disconnected or detached requires the ability to recognize the disjunctive aspects and order them in a way that reveals connections not directly apparent. Most students attain this ability by age 13 - 14.

Materials

2 story problems* (described belcw).

Task

The student (S) is sitting alongside or opposite the experimenter (E).

Procedure

E tells Story I to S and then seeks information concerning S's solution to the problem raised in the story. S is allowed to take notes and can ask for information to be repeated.

Story I

I hiked with my brother in the mountains. When we arrived at a cabin, we found that we had forgotten our provisions. We looked in the cabin and found some remainders of food, but the food was not fresh.

We found a little condensed milk, some soup, and some old preserved meat. My brother ate the meat and some soup. I ate some soup and the condensed milk. An hour later we were both sick to the stomach. One of the three foods made both of us sick. Which could it have been? Explain your answer.

Checking Procedure

E tells Story II to S and then seeks additional information concerning S's explanation.

Story II

Mr. Smith is the manager of an apartment house that has three garages in it. The first garage belongs to Mr. Smith, the second to Mr. Moss, and the third to Mr. Williams. There are two keys to each garage. For the first garage, one key belongs to Mr. Smith and the other to Mr. Williams. For the second garage, one key belongs to Mr. Moss and the other to Mr. Williams. For the third garage, one key belongs to Mr. Williams and one to Mr. Smith. One day it was found that some things were stolen from each of the garages. The police inspector said that whoever stole the items had keys to all the garages. Whom did the police inspector arrest? Explain your answer.

*The stories are adapted from: Morf, Albert. Les relations entre la logique et le langage lors du passage du raisonnement concret au raisonnement formel. In Apostol, L., B. Mandelbrot, and A. Morf. Logique, langage et theorie de l'information. *Etudes D'epistemologie Genetique, Vol. 3.* Paris: Presses Univ., 1957, 173-204.

Analysis

Careful listening to S's explanations will reveal that the type of justification given by an S who is at the pre-Disjunctive Reasoning Stage of development, is limited to simple intuitive and causal solutions. An S at the Disjunctive Reasoning Stage will take disjunction into consideration.

For example, a justification for Story I might be "*the milk*" or "*the meat made them sick because such foods are known to spoil and can cause sickness.*" Even though a pre-stage S may be capable of telling who ate what and may retain all the facts perfectly, he or she will seek and give concrete causal solutions and ignore one part of the information. In this example, the fact that the milk or the meat was eaten by only one person was neglected.

Similarly, a justification given for Story II might be "*Mr. Smith, because he is the manager*". In this justification, S focuses upon the occupation and intuitively feels that the manager must be the one who has access to all the garages. The pre-stage S never realizes that the manager does not have all the keys.

It is clear that pre-stage students will not make an attempt to work out solutions to these problems from the text. An intuitive or concrete situation in the story will dominate their explanations. Students in transition will often combine or confuse the causal and formal arguments in the same task.

E may also find that in the statement "*One of these three foods made both of us sick.*" in Story I either "*one of*" or "*both of us*" is suppressed by pre-stage students. This particular difficulty for the student seems to be in the combination or multiplication of three hypotheses with the phrase "*both of us.*" The pre-stage student's reasoning cannot encompass all the conditions.

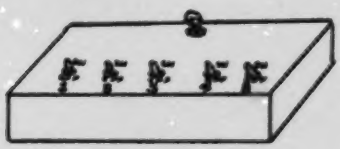
Combinatorial Reasoning

Principle

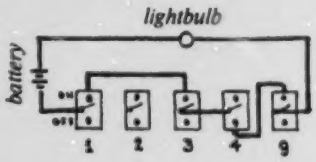
Solving problems that contain a number of possible solutions requires thinking that involves a complete combinatorial analysis. For example, combinatorial reasoning is necessary in order to determine all the possible five-card combinations that might be dealt from a standard deck of playing cards. Most students attain the ability to do this type of reasoning by age 14 - 15.

Materials

A small screen to keep five toggle switches from view.
A box made up of five toggle switches and a lightbulb.[1]



Switchbox



Wiring Diagram

Task

The student (S) is sitting opposite or alongside the experimenter (E).

Procedure

E shows S the box and tells S the light can be turned on by some combination of the switches in *on* and *off* positions.

E hides all the switches behind a screen and says: "*I am moving switches on and off behind this screen so as not to give you clues concerning how many of these switches, if any, one needs to move to make the light go on.*"

After clicking a number of switches, E says: "*When I move the last switch, labeled "g," the light will go on.*" E switches "*g*" and the light goes on.

E again clicks a number of switches behind the screen, then returns each switch to its *off* position. NOTE: Be sure the number of clicks does not provide a clue.

E: "*Now I want you to make the light go on by using switches 1, 2, 3, 4, and g as you wish. For each trial, decide ahead of time which switches you are going to use, and tell me what they are.*"

After S's first trial, E says: "*Return the switches you have just tried to the off position, making sure that switch g is the first one turned off.*" E allows S to proceed with other trials but makes sure that before another combination is tried, the switches are

[1] This task was adapted by permission from: Murray, Kenneth R. *Chemical yellow — an electronic version.* Paper presented at the University of California, Berkeley, 1971.

returned to the *off* position by always returning the *g* switch first.

E keeps track of all the combinations that S tries.

If S does not proceed in a systematic way or becomes frustrated, discontinue the task.

If S is successful, E asks a series of questions:

a. "*Which switches make the light go on?*"

b. "*What will happen if the switches are turned on in this order: 2, 1, 3, g?; 1, 3, g, 2?; 4, 1, 3, g?; 1, 3, g, 4?*"

c. "*What do switches 2 and 4 do?*"

d. "*Is there any other way you can make the light go on?*" (There are two possible combinations.)

Analysis

In order to turn on the light, other than by chance, the student must be equipped with a mental combinatorial system that enables him to keep the various permutations of possibilities in mind.

The student at Piaget's IIIa level of Formal Operational Thought will use a systematic approach to testing all possible combinations of the switches but will have some difficulty in answering the series of questions posed by the experimenter. The IIIb level student will have no trouble in answering the questions.

The pre-stage level student will not be able to test systematically all the combinations that are possible and will tend to explore possibilities in a partially systematic way or in a completely trial and error fashion.

Alternative Task I

Piaget's chemical substances interview[2] can be used to examine the extent of a student's ability to subject all the variables of a problem to combinatorial analysis. This interview follows the procedures and questions described above but uses five colorless, odorless liquids in place of the box with switches. The liquids used are: 1) diluted sulphuric acid; 2) water; 3) oxygenated water; 4) thiosulphate; g) potassium iodide.

This alternative task requires the student to produce the color *yellow* by using some combination of 1, 2, 3, 4, and g. When 1 + 3 + g or 1 + 2 + 3 + g are combined, the color appears. The water (2) has no effect when added to 1 + 3 + g. Thiosulphate (4) bleaches the mixture 1 + 3 + g or 1 + 2 + 3 + g.

Alternative Task II

See Workshop F, page 31, for another alternative procedure.

[2]Inhelder, B., and J. Piaget. *The growth of logical thinking from childhood to adolescence.* New York: Basic Books, 1958, 108-109.

Implicational Reasoning

Principle

Solving problems which contain a finite number of possibilities, each of which might be the solution, requires thinking that can systematically sort out the total number of possibilities, then combine the appropriate ones to formulate a solution. Most students attain this ability by age 14 - 15.

Materials

2 story problems* (described below).

Task

The student (S) is sitting alongside or opposite the experimenter (E).

Procedure

E tells Story I to S and then seeks information concerning S's solution to the problem raised in the story. S can take notes and can ask for information to be repeated.

Story I

Complaints were received at a watch factory, because there had been so many watches that had something wrong with them. The owner of the factory sent a man to study what was going on. The man examined the machinery that was used to make the watches and talked to the workers. He then telephoned the owner and said, "I have found that all the watches that were made in the month of September are faulty."

Later, the owner looked at some watches on his desk. He picked up the first watch and said, "This one was made in September; it has, therefore, a fault." Could the owner say that or not? Explain why.

The owner then looked at the second watch and said, "This was made in July, so I am sure that it doesn't have a fault." Could the owner say that or not? Explain your answer.

The owner next looked at the third watch, which he knew had a fault, and said, "This one has a fault, so I know it was made in September." Could he say that or not? Explain why.

Finally he picked up the fourth watch and said, "I know this one has not a single fault, so it was not made during the month of September." Could he say that or not? Explain.

Checking Procedure

E tells Story II to S, then seeks additional information concerning S's explanation.

*Stories are adapted from: Morf, Albert. Les relations entre la logique et le langage lors du passage du raisonnement concret au raisonnement formel. In Apostol, L.,B. Mandlebrot, and A. Morf. Logique, langage et theorie de l'information. *Etudes D'epistemologie Genetique, Vol. 3.* Paris: Presses Univ., 1957, 173-204.

Story II

A group of male students from all over California were given a trip to New York City. When they were about to leave California, the students who came from San Francisco decided that they would each wear a blue tie. While on the trip, two of the students from San Francisco were walking along a New York street and saw another student they knew was from California, but they couldn't see his tie because he was walking away from them. One of the students asked, "What color do you think his tie is?" What do you think his friend said? Explain.

Later they saw another student coming toward them wearing a blue tie. His friend asked, "Do you believe he is from San Francisco?" What was the reply? Explain.

Analysis

When the solution to problems involving implicational reasoning is faulty, the relationship between given facts is always neglected in part or inverted. Such distortions in thinking are typical of the pre-stage student. The distortions affect not only the answer given, but also the student's perception of the wording.

In the problem of the defective watches, the phrase "*all the watches that were made in September are faulty*" is the key to distinguishing formal operational thinking from pre-stage thinking. The phrase is the central part of a system of propositions: (a) made in September, (b) not made in September, (c) defective, and (d) not defective. Although pre-formal students will accept the phrase in its proper sense, they will at the same time convert it to "*all the defective watches were made in September.*" Thus the implication of the phrase is treated as mutual. Students at the Implicational Reasoning stage will not treat the implication as an equivalence.

Essentially, the problem involves two basic propositions about watches along with the negation of each: (1) either a particular watch was made in September, or it was not (i.e., was made in some other month); and (2) either the watch is defective, or it is not. The two propositions, or their negations, can be combined in four possible ways as illustrated by the following table.

		Was the watch made in September?	
		Yes	No
Is the watch defective?	Yes		
	No		

The statement of the problem gives some information as to which of the four possibilities actually exist. When the investigator reports that all watches made in September are defective, it means that the left-cells of the table can be filled in like this.

Is the watch defective?	Was the watch made in September? Yes	No
Yes	+	
No	o	

Here a "+" means that there exist watches about which a particular combination of propositions is true. A "o" means that there are no such watches.

Suppose that a watch is found that is *not* defective. The information given would enable one to conclude that this watch was *not* made in September. Thus another cell can be filled in on the table.

Is the watch defective?	Was the watch made in September? Yes	No
Yes	+	
No	o	+

But suppose another watch is found that *is* defective. Can one conclude that it was made in September? Certainly not. It might be a watch that belongs in the upper right-cell of the table.

The moral of the story is that, just because "*made in September*" implies *defective*, it does not automatically follow that *defective* implies "*made in September.*" One has to consider *all* of the possibilities.

Story II can be analyzed in a similar way by using a true-false table as described above.

Controlling and Manipulating Variables

Principle

Solving problems that involve the identification and systematic testing of each variable requires the ability to separate variables by exclusion. For example, factors that influence the rate of swing of a pendulum are determined by isolating possible factors and testing them one at a time, while all others are held constant. Many students and some adults are never able to identify and isolate the possible relationships involved in this and similar problems. Most students attain this ability by age 14 - 15.

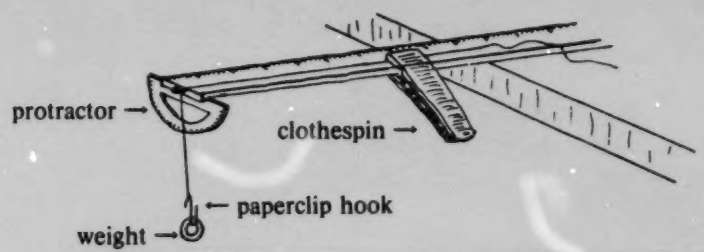
Materials

3 feet of thread.
1 sturdy yardstick with a small eyelet or U-nail driven in one end.
1 clothespin
1 paper clip
1 protractor
At least 6 identical washers or fishing weights.
Tape
Optional: stopwatch

Task

The student (S) is sitting alongside or opposite the experimenter (E).

E attaches the paper clip to the thread, suspends the thread from the yardstick, hangs two weights on the paperclip, and then sets the weights in motion.



Establishing the Variables

NOTE: In this problem there are at least four possible variables which can be tested: the length of thread, the weight, the angle at which it is released, the amount of force imparted when released.

E states: "*This is a pendulum and these are a number of different weights which can be attached to or removed from the thread.*" E demonstrates by removing one weight.

E: "*Notice that the thread can be made shorter or longer.*" E demonstrates how to adjust the length of the thread, then asks S to adjust it so that it is two inches long.

E: "*Notice that the pendulum can be released from different positions.*" E demonstrates how to release the weight using the protractor as a guide, then allows S to release it.

E: "*Notice that I can release the weight with more or less force.*" E demonstrates, then allows S to try.

Procedure

E says: "*See if you can find out what might make the pendulum swing faster or slower.*"

S tries something.

E probes to obtain insight into whether or not S is following a systematic plan: "*Tell me what you are doing.*"

After S has experimented with the pendulum and made some statement about its operation, E probes to determine the relationship or lack of relationship between the action of S, his observations, and inferences: "*Tell me why you think it is so.*"

Since it is important for E to establish the status of such factors as string length, weight, release height, and release force in S's thinking, E might ask S about the role of any factors that S has not mentioned. For example, if S does not mention the possible effect of different weights, E might ask: "*If we make a pendulum using more (fewer) weights, will the pendulum swing faster or slower or the same?*" E might also arrange an untried experimental situation and ask S to make a prediction.

Analysis

If S is at the Controlling and Manipulating of Variables Stage of development, he will reveal concern about the possible interrelationships of variables and therefore will attempt to control as many as possible while varying only one. If S is not at that stage, he might not consider the importance of controlling variables (e.g., S might explore the effect of increasing the weight of the pendulum but not consider it important to keep the length of the thread or the height of the first swing constant during his experiments).

The Controlling and Manipulating of Variables Stage student will be one who is able to perform controlled experiments where single variables are isolated in turn to study their effects. The student will also be able to report systematically all of the combinations that are possible. Furthermore, he will demonstrate his capacity for making hypotheses before he experiments. Instead of being closely bound to immediate experience, he can suggest "*What might happen if...*" and then perform the actions necessary to confirm or disprove his suppositions.

If S is not at that stage of development, he may be able to make accurate observations, investigate potential determinates, and might conclude that weight has an effect. However, he will not be able to design a controlled experiment and will not be able to provide a correct conclusion.

If S is at the pre-stage, he will try to solve the task in a haphazard, random way. He will not observe accurately, and his conclusions will be unrelated to the evidence.

Alternative Procedure

See Workshop G, page 35, for an alternative procedure.

Deductive Reasoning

Principle

Deductive reasoning moves from the more general to the less general. A deductive argument or inference is one that is logically conclusive (i.e., its validity or consistency can be certified by logical considerations alone, usually through a transitivity of implications). For example:

All women are mortal.

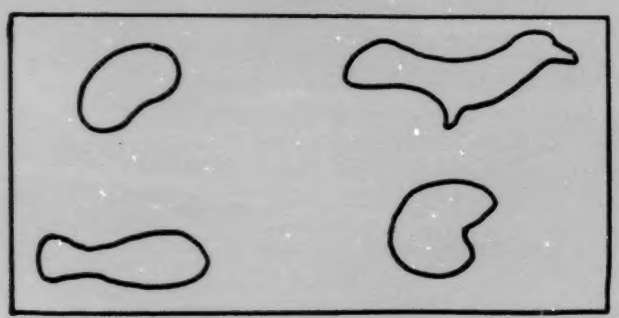
All queens are women.

Thus all queens are mortal.

The conclusion of a deductive argument is simply an explicit statement of something that is implicit in the premises. Many students (50-60%) attain the first substage level (IIIa) of this ability by age 15 - 17. Only a few (10-15%) reach the second substage (IIIb).

Materials

Consumable map showing four islands:



Task*

The student (S) is sitting opposite or alongside the experimenter (E).

E places the map on the table in front of S and asks S to examine it for a moment.

Procedure

E provides information: "*This is a map of four islands in the ocean. The islands are called Bean Island, Bird Island, Fish Island, and Snail Island. You may make notes or marks on your map to help you remember the names and other information. You may ask questions about the information at any time.*"

*This task was adapted by permission from: Karplus, Elizabeth F., and Robert Karplus. Intellectual development beyond elementary school I: deductive logic. *School Science and Mathematics*, 70, 5, 1970, 398-406.

E continues: "*People have been traveling among these islands by boat for many years, but recently an inter-island airline started in business. Listen carefully to the clues I give you about the possible airplane trips. The trips may be non-stop between islands, or they may include stops at one or more islands on the way. When I say a trip is possible, it can be made in both directions between the islands. In other words, the airline has established the routes in both directions.*"

E might indicate that the islands are so named because of their distinctive shapes. E should be sure S is listening and is ready to hear the clues.

E: "*First Clue: People can go by airplane between Bean and Fish Islands.*"

E: "*Second Clue: People cannot go by airplane between Bird and Snail Islands.*"

E: "*Use the two clues I've given to answer this question: Can people go by plane between Bean and Bird Islands?*"

After S responds, E asks S to explain his or her answer.

E: "*Third Clue: People can go by plane between Bean and Bird Islands.*"

E: "*Use all three clues to answer these questions: 1) Can people go by plane between Fish and Bird Islands? 2) Can people go by plane between Fish and Snail Islands?*"

After each question, E allows S to respond and asks S to give a rationale. NOTE: E may repeat clues as often as requested by S.

Analysis

A student is at the pre-stage of development if his or her explanation makes no reference to the clues and/or introduces *new* information. A pre-stage student might answer question 2: "*Yes, because there are flights.*" or "*No, because the plane can run out of gas and go down in the water.*"

Another indication that a student is at the pre-stage is if his or her explanation uses the clues to construct a concrete (symbolic) model or models that are then used to make predictions. A pre-stage student, using as a reference a model(s) he or she has drawn, might answer question 1: "*Can't tell because Bean Island has an airport, but Bird Island might not have an airport.*" or question 3: "*No, Snail Island must be the one with no airport, so people from Fish Island can't get there.*"

NOTE: The model-based approach, when correctly used, leads to correct answers; but within the framework of concrete operations, the use of the airport model assumes information not given in the clues and will not generalize to solve different puzzles with different data.

Proportional Reasoning

Principle

Proportional reasoning is the comparative relation of one thing to another in part or whole and is expressed in terms of magnitude, quantity, or degree. For example, ratio can be applied to solve a problem of comparing two different units of length using two objects of different lengths. A student with the ability to operate with ratio, using one-inch units of length to measure lengths of one inch and three inches respectively, can predict mathematically that if it takes two one-half inch units to equal the shorter length (one inch), then it will take six one-half inch units (three times as many) to equal the longer length. Most students attain this ability by age 16 - 18.

Materials

1 chain of #1 Gem paper clips (10 clips long).
1 chain of jumbo-sized Gem paper clips (10 clips long).
1 worksheet depicting a drawing and statements (See Figure of Mr. Short).
1 display chart depicting Mr. Short.
Several straight pins.
1 chalkboard or other writing surface including writing implement.
1 pencil for the student.

NOTE: On the worksheet Mr. Short should be exactly four chained jumbo-sized and six chained regular paper clips tall. The picture of Mr. Short on the display chart should be the same size.

Task*

The student (S) is sitting opposite or alongside the experimenter (E).

E gives S the worksheet and the chain of jumbo-sized paper clips.

E has a chain of #1 (regular size) paper clips.

Procedure

E shows S the display chart depicting Mr. Short.

E: "*This problem is about Mr. Short, who is just like the man on your paper, and Mr. Tall, who is similar to Mr. Short, but larger. I have his picture back at my office, so I can't show him to you.*"

E displays chart illustrating Mr. Short.

E: "*I will measure how high Mr. Short is with my chain of big paper clips—we'll call these clips 'biggies.'*"

E displays the chain of clips.

*This task was adapted by permission from: Karplus, Robert, and Rita Peterson. Intellectual development beyond elementary school II: ratio, a survey. *School Science and Mathematics*, 70, 9, 1970, 813-820; Karplus, E.F., R. Karplus, and W. Wollman, Intellectual development beyond elementary school IV: ratio, the influence of cognitive style. AESOP, Lawrence Hall of Science, University of California, Berkeley, 1973.

E: "*We'll write down the results of the measurements on the board.*" E writes "*Mr. Short*" and "*Mr. Tall*" on the board.

E: "*How high is Mr. Short?*" (four biggies)

E hangs his paper clip chain from a pin at the top of Mr. Short's head. S counts the number of clips and responds verbally. E writes the number on the chalkboard.

E: "*When I measured Mr. Tall in the same way, I found he was six biggies high.*" E writes the number on the board.

E sets down the chart and puts away the big paper clips. These are not used again.

E: "*Now I would like you to do three things using your worksheet. First, measure Mr. Short using your small paper clips—we'll call the clips 'Smallies.'*"

After S measures, E says: "*Second, I want you to predict the height of Mr. Tall if you could measure him with smallies. Third, I want you to explain how you make your prediction. Explain as best you can how you decided the number of smallies in your prediction.*"

NOTE: S's questions or requests for more information are to be referred to data on the chalkboard. The measurements are not repeated after the display chart is put aside, nor is S given access to the big paper clips. E tells S that different people answer the questions differently, that there is no one correct way to solve the problem, that S can measure in his own way, and that great accuracy is not important. About 10 - 15 minutes are required to complete this task.

Analysis

A student is at the pre-Proportional Reasoning Stage of development if his or her explanation refers to estimates, guesses, appearance, or extraneous factors without using the data or if he or she uses the data in an illogical way. A pre-stage student might answer: "*Six, because that is what I think it is.*"

A student is at the pre-stage if his or her explanation uses proportionality, but is an *estimation* of length ratio instead of a computation of it from the data. A pre-stage student might answer: "*Six. I think that one biggie equals one and one-half smallies, so I took four biggies and added them up and got six.*"

A student is at the Proportional Reasoning Stage if he or she uses proportionality and makes clear how the ratio is derived *from the measurements* on the two figures. The student may or may not use the word *ratio.* A student might answer: "*Six. The ratio of the smallies is 2:3 (two to three) so you figure the big paper clips would* also *have the ratio 2:3.*"

NOTE: The fact that the numerical result given by each of the above students is correct, is not as important in determining the developmental level as the substance of the student's explanation of what he did.

Alternative Procedure

See Workshop H, page 37, for an alternative procedure.

DIRECTIONS FOR PREPARING THE WORKSHEET AND DISPLAY CHART FOR PROPORTIONAL REASONING TASK

1. Make worksheet copies of the figure and the accompanying statements on the next page. The figure is Mr. Short. Be sure your copies of Mr. Short are exactly the same height as the figure (six chained #1 Gem paper clips high or four chained jumbo-sized Gem paper clips high). Use a worksheet with each student interviewed.

2. Copy the figure without the statements. Place the copy on the front of a display chart.

   NOTE: Reproduction of the figure sometimes changes the size of the figure. Be sure the figure you use is six chained #1 Gem paper clips high.

Mr. Short

How tall is Mr. Short when measured with small paper clips?

Predict the height of Mr. Tall when measured with small paper clips.

Explain how you figured out your prediction.

self-directed workshops

The following pages
contain several self-directed workshops.

Except for the first workshop,
the order in which they are done,
the number that are done,
and the time it takes to do them are up to you.

workshop **A**

developing the ability to administer propositional reasoning tasks

The purpose of this personal workshop is to guide you in practicing the tasks shown in the previous section. After finishing this workshop, you should be able to administer the tasks and record, in a variety of ways, the responses of students.

Study the guidelines on the following page and then practice administering any two of the propositional tasks with a friend. Consciously incorporate some of the ideas from the guidelines, as you practice.

Because the responses of students may be very complex, it might be helpful to use some type of a record-keeping system. The two pages following the guidelines show a few ways in which information can be easily recorded in chart forms.

Use the space below to jot down personal notes about your practice experiences.

GUIDELINES FOR ADMINISTERING PROPOSITIONAL TASKS

The Cognitive Operations

Propositional tasks involve combinations of cognitive abilities. For each task it is important that you find out if the student has the ability to perform two at a time, three at a time, and so forth. Sixteen cognitive abilities, identified by Piaget, are listed on page 48. Most of these are included within the tasks presented in the previous section.

The Assessment Method

The assessment method is a consistent and powerful tool. Essentially, it is the careful observation of a student's reaction to an interaction with some aspects of the physical world such as the swing of a pendulum, the balance of a scale, or the motion of an object.

When you use the method, it is important to remember:

a. that the purpose is to determine how students approach or go about solving a task, rather than whether they "get the right answer."

b. that when you interact with students, manipulate the materials with them, but let students' responses and actions guide the discussion.

c. that you must approach students at their individual levels, observing what they do and say, and noting how they respond to situations. Do this independently of any preconceived expectations you might have concerning how they should act or respond.

d. that the method will not arm you with a set of predetermined questions.

e. that the results will not produce standardized responses to adult questions. The results will simply indicate something about each student's interaction with his or her view of the world.

Use of the assessment method will point out that there are very few incorrect responses when one asks students questions, for responses usually are correct in terms of the intellectual framework in which they are given. Once this concept is understood and accepted, the whole approach to educating can become more effective.

Sample Record Form for Probabilistic Reasoning Task

Directions: Record action and verbal response.

Procedure	Student's Actions		Student's Verbal Responses	Comments
	blue	white		
I_1		√	Because there are more whites.	
2	√		Because there are more blues.	
II_1	+5		Makes the amounts even.	
2	-5			
III_1	+3			
2		+3		
3				
4				
5				
IV_1				

Sample Record Form
for Combinatorial Reasoning Task

Directions: Record order of switches turned on.

Trials	Switches					Comments
	1	2	3	4	g	
1	1	2	3	4	5	
2	5	1	2	3	4	
3	4	5	1	2	3	
4						
5						
6						
etc.						

workshop **B**

assessing the administration of propositional tasks

The purpose of this workshop is to help you assess your ability to administer a task using the assessment method.

Prepare an audio- or videotape of yourself administering a propositional task to a student.

At your leisure, use the form on the next page to guide an assessment of your use of the method. You might wish to discuss various points with colleagues.

Self-Assessment: Using the Assessment Method	
Assessment Considerations	Personal Notes
a. What aspects of the interview indicated that I was non-judgmental and accepting of the student's ideas?	
b. What examples in the interview indicated that I avoided leading the student toward an idea I had?	
c. What evidence is there that I let the student's responses and actions guide the discussion?	
d. What evidence indicates that I approached the student at his or her level and responded to the situation independently of adult expectations of how he or she should act?	
e. What evidence indicates that I formulated hypotheses concerning the student's strategies and tested the hypotheses through manipulations of the materials?	
f. In what ways did I check for justifications?	

workshop **C**

administering propositional tasks to a student

The purpose of this workshop is to guide you in assessing the ability of one student. After doing this workshop, you should be able to identify a student's level of ability.

Select a student in your classroom about whom you wish to learn more. Administer one or more propositional tasks.

At your leisure, study the student's responses in terms of the analysis given for each propositional task you used. Keep a record, diary, or profile of the assessment. Decide what kinds of experiences would be suitable in a learning experience for this student.

You may wish to audio- or videotape this experience, so that you can discuss your assessment with a colleague.

workshop **D**

administering propositional tasks to students at different levels

The purpose of this workshop is to let you compare different levels of responses to the same experience. After doing this workshop, you should be able to better identify the levels of ability of different students.

Select one task and administer it to three different students: one primary, one intermediate, and one high school.

At your leisure, compare the students' responses in terms of the analysis given for the task you selected.

You may wish to audio- or videotape this experience, so that you can discuss the information with a colleague.

workshop **E**

observing the administration of propositional tasks

The purpose of this workshop is to provide additional practice in assessing levels of development and in analyzing interviewing procedures. After finishing this workshop, you should be able to better identify levels of abilities.

View the PDARC videotape on Propositional Reasoning* with several other teachers. The tape will show students at different levels of cognitive development responding to a propositional ability task.

When you finish viewing the tape, discuss the following with your colleagues:

1. What clues were observed that indicated the stage of development for each student interviewed?
2. Compare the responses of the students on this tape with responses of those you have interviewed (e.g., in workshop B).
3. Did the interviewer use the assessment method throughout each interview? (You might use the guide given in workshop B to check the interviewer's procedures).

*Videotape available from PDARC, Department of Education, 1649 Tolman Hall, University of California, Berkeley, CA 94720

workshop **F**

analyzing student's thinking:

combinatorial reasoning

The purpose of this workshop is to give you some practice in recognizing stage differences in the ways students resolve combinatorial reasoning problems.

The following problem was developed by Professor Anton Lawson, University of Arizona* and is reproduced here with his permission. Read the problem, then work out an answer to it before you turn the page.

*Lawson, Anton E. and Renner, John W. "Piagetian theory and biology teaching," *American Biology Teacher* 1975, 37, 336-343.

THE ALGAE PUZZLE

A population of crabs which eats algae lives on a seashore. On the seashore there are four kinds of algae: yellow, red, brown, and green algae.

Yellow—Y

Red—R

Brown—B

Green—G

Dr. Saltspray, a biologist, is interested in determining which of the types of algae are actually eaten by the crabs. He plans to find out by examining the stomach contents of the crabs. Before he does his investigation he lists all the combinations of algae he thinks possible to find in the stomachs. Write down each possible combination of algae he can find. Use letters Y, R, B, and G to save space.

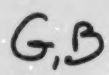
_____

_____

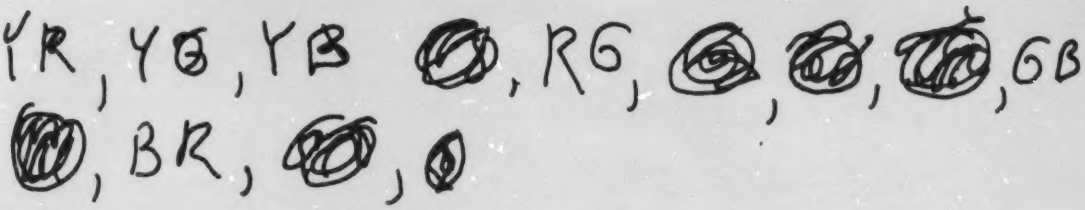
_____

The following are samples of responses collected by Professor Lawson. The responses have been grouped as Student A responses and Student B responses. All Student A responses are typical of students who are not yet at the Combinatorial Reasoning level while all Student B responses are at or beyond the level. Discuss with colleagues the characteristics of Student A responses and how they compare to the characteristics of Student B responses.

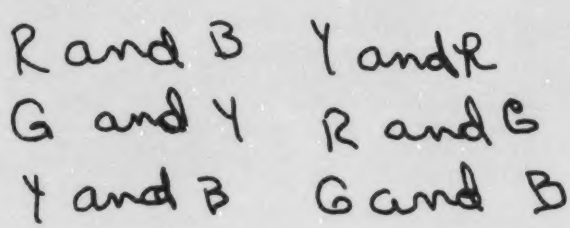
Student A_1 (High School Senior)



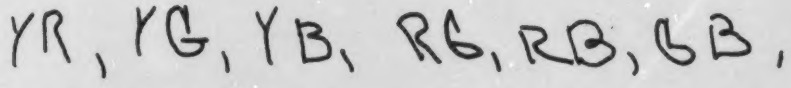
Student A_2 (High School Sophomore)



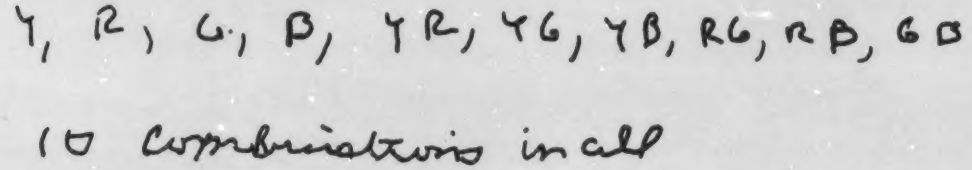
Student A_3 (College Junior)



Student A_4 (College Freshman)



Student A_5 (College Graduate)



Student B_1 (High School Junior)

4RGB, 4G, 4B RGB,
4GB, GR, GB, BR,

Student B_2 (College Freshman)

Y, R, G, B R, G, B G, B B
Y, R G R, G G
Y, R R
Y

Student B_3 (College Freshman)

Y, R Y, R, G R, G 11 different combinations
Y G Y, G, B R B
Y, B Y, B, R R G, B
G, B
Y R G B

Student B_4 (College Junior)

YR, YG, YB, YRG, YGB, RGB,
RG, RB, GB, YRGB, Y, R, G, B

Student B_5 (High School Junior)

YRGB BGRY GRYB RYBG

YR RY YRG
YG GY YRB
YB BY YBG
RG GR YBR
RB BR
GB BG
BGR RGB
BGY YGB

Student B_6 (High School Freshman)

YR, YG, YB, RG, RB, GB.
YRG, YRB, YGB, RGB, YRGB,
Y, R, G, B.

Student B_7 (College Sophomore)

Y, R, G, B, YRGB, YR, YG, YB, RG, RB, GB,
YRG, YRB, YGB; RGB.

Student B_8 (College Sophomore)

YRGB	RGB	GB	B	YGB
YRG	RG	G		YB
YRB	RB			YG
YR	R			
Y				

workshop **G**

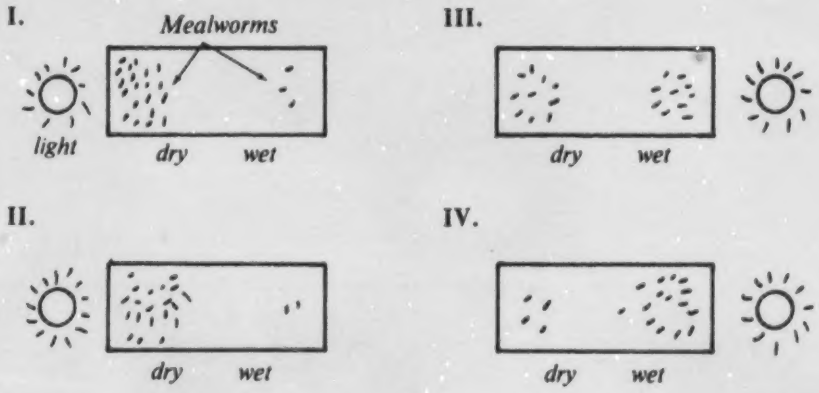
analyzing student's thinking: controlling and manipulating variables

The purpose of this workshop is to give you some practice in recognizing stage differences in the ways students resolve problems requiring the control and manipulation of variables.

The following problem was developed by Professor Anton Lawson, University of Arizona* and is reproduced here with his permission. Read the problem, then work out an answer to it before you turn the page.

THE MEALWORM PUZZLE

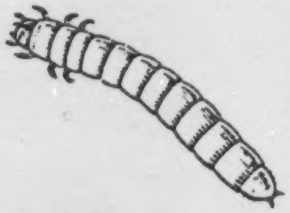
An experimenter wanted to test the response of mealworms to light and moisture. To do this he set up four boxes as shown in the diagram below. He used lamps for light sources and constantly watered pieces of paper in the boxes for moisture. In the center of each box he placed 20 mealworms. One day later he returned to count the number of mealworms that had crawled to the different ends of the boxes.



The diagrams show that mealworms respond to (response means move toward or away from):

A) light but not moisture.
B) moisture but not light.
C) both light and moisture.
D) neither light nor moisture.

Please explain your choice.



*Lawson, Anton E. "Developing formal thought through biology teaching," *American Biology Teacher* 1975. 37, 411-429.

The following are typical student responses to the Mealworm Puzzle. Read these responses and compare them with your own. Again, look for similarities and differences between type A and type B responses.

Student A_1 (College Junior)

"No definite pattern was followed by the mealworms."

Student A_2 (College Freshman)

"Because even though the light was moved in different places the mealworms didn't do the same things."

Student A_3 (College Freshman)

"They usually went to the end of the box with the light."

Student A_4 (High School Sophomore)

"Because there are 17 worms by the light and there are only 3 by the moisture."

Student B_1 (College Sophomore)

"Boxes I and II show they prefer dry light to wet and dark, Box IV eliminates dryness as a factor, so they do respond to light only. Box III shows that wetness cancels the effect of the light, so it seems they prefer dry. (It would be clearer if one of the boxes was wet-dry with no light)."

Student B_2 (High School Senior)

"When the light was on the dry side they all crowded to the dry side. When it was on the wet side, an equal amount went to each side."

Student B_3 (High School Freshman)

"In experiment 3 the mealworms split 1/2 wet, 1/2 dry. So it's safe to assume that light was not the only factor involved."

Student B_4 (College Freshman)

"The mealworms in all cases respond to light. However, in box 3 the division is about 1:1. This shows that the worms are attracted to the light but not like the situations where the dry area was next to the light. When there is no choice between wet and dry such as in case IV the worms turn to the light. Note: We might also test a box like this with no light to further verify the effect of moisture.

workshop **H**

analyzing students' thinking:

proportional reasoning

The purpose of this workshop is to give you some practice in recognizing stage differences in the ways students resolve problems requiring proportional reasoning.

The following problem was developed by Professor Anton Lawson, University of Arizona* and is reproduced here with his permission. Read the problem, then work out an answer to it before you turn the page.

THE FROG PUZZLE

Professor Thistlebush, an ecologist, conducted an experiment to determine the number of frogs that live in a pond near the field station. Since he could not catch all of the frogs he caught as many as he could, put a white band around their left hind legs, and then put them back in the pond. A week later he returned to the pond and again caught as many frogs as he could. Here is the professor's data.

First trip to the pond
55 frogs caught and banded

Second trip to the pond
Of 72 frogs caught,
12 were banded

Total number of frogs in the pond

_____

The Professor assumed that the banded frogs had mixed thoroughly with the unbanded frogs, and from his data he was able to aproximate the number of frogs that live in the pond. If you can compute this number, please do so. Write it in the blank above, and in the space below explain in words how you calculated your result.

_____

_____

_____

*Renner, John W. and Lawson, Anton E. "Piagetian theory and instruction in physics," *The Physics Teacher*, 1973, 11, 165-169.

This page presents student solutions to the Frog Puzzle. Please read the responses and compare them with yours. Look for key differences between Student A responses and Student B responses.

Student A_1 (College Senior)

Answer: 72
Explanation: "It would be 72 because that is all he was able to catch."

Student A_2 (College Freshman)

Answer: 115
Explanation: "55 were caught and banded, 55 with bands. He didn't catch every frog so 55 and 60 would be 115."

Student A_3 (High School Sophomore)

Answer: 115
Explanation: "I say 115 because 55 + 60 = 115 frogs."

Student A_4 (College Junior)

Answer: about 200
Explanation: "60 frogs were caught. I'd have to try a third time (take a second sample). How big is the pond? Add 60 and 55 together. It's poor experimental data. I wouldn't try to make a guess without a third try. More would come up in a third try. I'd guess 200."

Student B_1 (College Sophomore)

Answer: 275
Explanation: "72 - 12 = 60 were not banded. 55/x = 12/60, x = 275 so the number of frogs is 275."

Student B_2 (College Junior)

Answer: 330
Explanation: "1/6 were banded. 55 x 6 = 330. What this implies is that one out of six mingled with the total frogs."

Student B_3 (College Sophomore)

Answer: 330
Explanation: "55:x :: 12:72
12x=3960
x = 330"

Student B_4 (College Freshman)

Answer: 330
Explanation: "You have to assume that in the week between the first and second sampling that none of the banded frogs died or were born. Also, the assumption must be made that the frogs mingled thoroughly. This may not be the case, but anyway, if you make all these assumptions the problem is simple. 12/72 = 1/6 so 1/6th of the frogs have bands 55 x 6 = 330."

workshop **I**

solving problems involving propositional reasoning

> *The purpose of this workshop is to give you an opportunity to do tasks which involve propositional abilities and to let you compare your thinking with an analysis of the logic involved. After this workshop, you should have an appreciation for the complexity of the propositional abilities.*

Listed below are two problems. Solutions to these problems require several cognitive operations at the propositional thinking level.

Try solving one or both of the problems. Work on the problem(s) long enough to give them a *good* try. If you have trouble with one of them, put it aside for awhile, then return to it and try again.

If you solve the problem(s), list the steps one would go through to show another person how to solve it. This procedure will reveal to you the combinatorial aspect of propositional thinking.

If you do not solve the problem(s) (they are really tough!), do your best, and then follow the analysis on the following pages. See the Appendix for additional propositional thinking problems.

1. *The Five Suspects.* Five suspects, A, B, C, D, and E, were rounded up in connection with a murder. Their statements were as follows:

A: "C and D are lying."
B: "A and E are lying."
C: "B and D are lying."
D: "B and E are lying."
E: "A and B are lying."

Who is lying?

2. *The Twelve Coins.* Someone gives you twelve coins (e.g., all quarters, all dimes). One coin in the set is counterfeit, and the other eleven are identical in every characteristic. The counterfeit coin is different only in the characteristic of weight—it is either heavier or lighter than the other coins.

Using an equal-arm balance scale, determine in no more than *three* weighings, which of the twelve coins is counterfeit *and* tell whether it is heavier or lighter than the other coins.

Good luck!

ANALYSIS OF THE FIVE SUSPECTS PROBLEM

For clarity, the following matrix describes the statements and claims made by the five suspects.

Claim

Statement	A	B	C	D	E
A			√	√	
B	√				√
C		√		√	
D			√		√
E	√	√			

In the explanation, the following notation is used:

A: Person A is telling the truth.

$\bar{A}$: Person A is lying.

$\rightarrow$: Implies.

$\wedge$: And (juxtaposition and ' is also used).

v: Or (either one or the other or both).

$\leftrightarrow$: Is equivalent to.

$\therefore$: Therefore.

To solve the five suspects problem, let us assume that A is telling the truth and see where this assumption leads.

1.0 $\bar{C} \wedge \bar{D}$, an immediate consequence of the assumption that A is telling the truth.

1.1 $\bar{C} \rightarrow B \text{ v } D \rightarrow B$, since $\bar{D}$, by 1.0

1.2 $B \rightarrow \bar{A} \wedge \bar{E}$, but $\bar{A} \wedge \bar{E}$ contradicts the assumption that A is telling the truth.
$\therefore$ our original assumption of A was wrong.

Now we can start again using our new knowledge:

$\bar{A}$ is the case.

2.0 $C \vee D$, since $\bar{A}$ is assumed.

2.1 $C \rightarrow \bar{B} \wedge \bar{D}$

2.2 $\bar{B} \rightarrow A \vee E$, but A contradicts the conclusion that $\bar{A}$ is the case. $\therefore$ E must be true.
By assuming C in 2.1, we conclude $\bar{A}$, $\bar{B}$, C, $\bar{D}$, and E:

$\bar{A}$: A is lying.
$\bar{B}$: B is lying.
C: C is not lying (i.e., C is telling the truth).
$\bar{D}$: D is lying.
E: E is not lying (i.e., E is telling the truth).

To check the consistency of this tentative solution:

$\bar{A} \rightarrow \overline{CD} \vee \bar{C}D \vee C\bar{D} : \overline{CD}$ is the case.
$\bar{B} \rightarrow \overline{AE} \vee \bar{A}\,E \vee \bar{A}\,E : \bar{A}\,E$ is the case.
$C \rightarrow \bar{B}\,\bar{D}$, which is the case.
$\bar{D} \rightarrow \overline{BE} \vee \bar{B}\,E \vee B\bar{E} : \bar{B}\,E$ is the case.
$E \rightarrow \bar{A}\bar{B}$, which is the case.

To guarantee that we have found the solution, we must check the alternative at 2.1, that D is true. is true.

2.1' $D \rightarrow \bar{B} \wedge \bar{E}$

2.2' $\bar{B} \rightarrow A \vee E$

Since $\bar{A}$ has been concluded, E follows; in contradiction to the conclusion of 2.1'. $\therefore$ the assumption of D was wrong.

Therefore, the solution that A, B, and D were lying is obtained.

ANALYSIS OF THE TWELVE COINS PROBLEM

In the analysis, the following notation is used:

I, II, III	=	The three weighings.
1 through *12*	=	The twelve coins.
L and *R*	=	The sides of the scale on which the coins are placed (Left and Right).
l, r, b	=	The results of a given weighing (l = left side goes down; r = right side goes down; b = the scale balances).

Weighing I

Place *1, 2, 3, 4, L* and *5, 6, 7, 8 R.*

If *Ib*	(Then the counterfeit coin is either 9, 10, 11, or 12). Go to weighing *Ib II.*
If *Ir*	(Then either 1, 2, 3, 4, is light or 5, 6, 7, 8 is heavy). Go to weighing *Ir II.*

Weighing Ir II

Place *1, 2, 3, 4 L* and *3, 6, 9 R.*

If *Ir IIb*	(Then either 4 is light, or 7 or 8 is heavy). Go to Weighing *Ir IIb III*
If *Ir IIr*	(Then either 1 or 2 is light, or 6 is heavy). Go to Weighing *Ir IIr III*
If *Ir III*	(Then either 5 is heavy, or 3 is light). Go to Weighing *Ir III III.*

Weighing Ir IIb III

Place *4, 8 L* and *9, 10 R*

If *Ir IIb IIIb*	(Then 7 is it, and it is heavy).
If *Ir IIb IIIr*	(Then 4 is it, and it is light).
If *Ir IIb III1*	(Then 8 is it, and it is heavy).

Weighing Ir IIr III

Place *1, 6, L* and *10, 11 R.*

If *Ir IIr IIIb*	(Then 2 is it, and it is light).
If *Ir IIr IIIr*	(Then 1 is it, and it is light).
If *Ir IIr III1*	(Then 6 is it, and it is light).

Weighing Ir III III

Place *5 L* and *9 R*.

If *Ir III IIIb* (Then 3 is it, and it is light).
If *Ir III IIIr* (Not a possible result).
If *Ir III IIII* (Then 5 is it, and it is heavy).

Weighing Ib II

Place *9, 10 L* and *11, 1 R*.

If *Ib IIb* (Then 12 is the counterfeit coin).
If *Ib IIr* (Then either 11 is heavy, or, 9 or 10 is light). Go to Weighing *Ib IIr III*.
If *Ib III* (Then either 11 is light, or, 9 or 10 is heavy). Go to Weighing *Ib III III*.

Weighing Ib IIb III

Place *12 L* and *1 R*.

If *Ib IIb IIIb* (Not a possible result).
If *Ib IIb IIIr* (Then 12 is light).
If *Ib IIb IIII* (Then 12 is heavy).

Weighing Ib IIr III

Place *9, 11 L* and *1, 2 R*.

If *Ib IIr IIIb* (Then 10 is it, and it is light).
If *Ib IIr IIIr* (Then 9 is it, and it is light).
If *Ib IIr IIII* (Then 11 is it, and it is heavy).

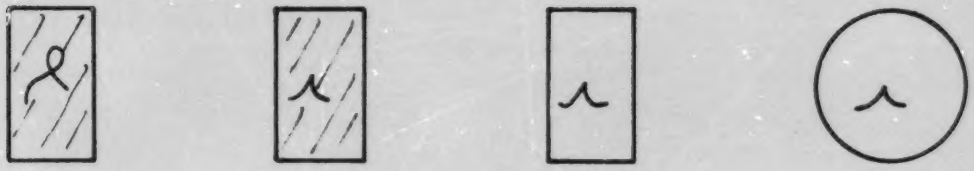
Weighing Ib III III

Place *9, 11 L* and *1, 2 R*.

If *Ib III IIIb* (Then 10 is it, and it is heavy).
If *Ib III IIIr* (Then 11 is it, and it is light).
If *Ib III IIII* (Then 9 is it, and it is heavy).

And now, here are three more problems for your analysis.

1. *Funny Money.* Some countries indicate the value of money by making different values different colors. Some show value by making each value a different shape. Others put different numbers on the money. A few use all three systems combined in some fashion. Assume that you could trade in as many of the foreign coins—shown below—as you wanted and that you would receive American money in exchange. What is the minimum number of coins you could trade in to see whether the color of a coin affects its value? Circle the coins which you would trade.



#1 Blue | #2 Blue | #3 Red | #4 Red

2. *Up in Smoke.* The Lowtar Cigar Company puts cardboard shapes in each of its packages. Each shape has a color on one side and a word on the other. If you get a package in which every shape with the color red on one side has the word "Low" on the other, you win a prize. What is the minimum number of cards you would have to turn over to see whether the set below is a winning one? Circle the one(s) which you would turn over.

#1 Red | #2 Blue | #3 | #4

3. You visit an island where two tribes live. One tribe always tells the truth. The other always lies. The truth tellers live on the northern side of the island, and the liars live on the southern side. Your problem is to determine who tells the truth by asking one native only one question.

   Suppose upon seeing a native walking in the distance, you tell a nearby native, "*Go ask that native in the distance which side of the island he lives on.*" When the messenger returns he answers, "*He says he lives on the western side of the island.*" Is the messenger a truth teller or a liar? How can you be sure?

workshop **J**

recognizing propositional reasoning in English

The purpose of this workshop is to have you transfer some of what you have learned to another field - English. After the workshop, you should be able to use your assessment skills more broadly.

Being able to *read between the lines* and to comprehend *higher* levels of meaning beyond the printed words are indicators of an advanced level of thought.

The following is a list of several literary sayings that a teacher might give to students and ask them to explain what each saying means. Below each saying is a possible statement given by a student. After you read them, decide whether or not the student's statement is at the Propositional Reasoning Stage of development. Discuss your analysis with colleagues.

1. KITES RISE HIGHEST AGAINST THE WIND - NOT WITH IT.

   What this really means to me is that it flies in the sky with the wind behind it.

2. IT OFTEN SHOWS A FINE COMMAND OF LANGUAGE TO SAY NOTHING.

   This means you talk too much.

3. IF THERE IS NO WIND - ROW!

   Once upon a time there was a man, and the man had a sailboat. One day the man thought to himself and said, "What a beautiful day for sailing." So he went and got his gear and went sailing. So here he was going along just fine until the wind stopped blowing, and he said to himself, "If there is no wind - row!" So he rowed back to his home and lived happily ever after.

4. CLOUDS - THE ONLY BIRDS THAT NEVER SLEEP.

   They are like birds in the sky, flying. They give you water and take it away. Sometimes they are black and sometimes white.

5. A FRIEND IS A PRESENT YOU GIVE YOURSELF.

   This means a lot to me. It reminds me of Bonnie. I like Bonnie a lot. And it means that it isn't a birthday present. It's a present of love and happiness.

REVIEW OF SOME RESEARCH ON PROPOSITIONAL ABILITIES

Prologue

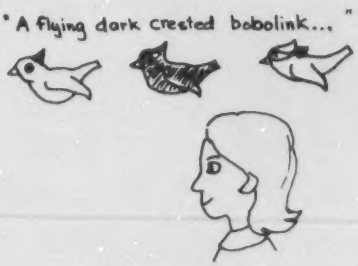
Propositional abilities constitute the basis for the most mature or advanced level of cognitive development. This level, described by Piaget as the Formal Operational Stage of cognitive development, generally develops throughout adolescence.

As the abilities develop, thinking begins to deal with statements or ideas about raw data, rather than with the raw data itself. The student might first use concrete operations of an earlier level of development to organize reality into classes or to order it in some way, but he or she then proceeds to generate propositions that use the organizations and to operate on the propositions via such abilities as conjunction, disjunction, implication, negation, compensation, and equivalence. Because it employs operations upon operations and the resultant statements about statements, propositional thinking is considered to be second-level thinking (Flavell, 1963).

To explain his notions of this stage of thinking, Piaget developed two models that, when combined, constitute an integrated cognitive structure. The first is a general model that sets forth the possible propositional abilities that underlie this level of thinking. The second is a specific model that describes the cognitive operations derived from the abilities. There is some debate about the validity of this model. It is presented here so that the debate can be more clearly understood.

a. The General Model

Theoretically, in approaching a problem a student at the Formal Operational Stage of development can scan the realm of the possible, begin to test empirically the validity of the possibilities, and can reach some conclusions about the results. The following analysis of a problem in which multiple variables must be considered will show the abilities that underlie the structure of this general model.



Suppose the solution of a problem required isolating the variables that cause a pendulum to swing rapidly. Considering only the variables of (1) the weight attached to the pendulum (w = heavy; $\overline{w}$ = light) and (2) the length of the pendulum (l = long; $\overline{l}$ = short), the following four combinations of variables are possible:

$(w \cdot l)$	$(\overline{w} \cdot l)$
heavy and long	light and long
$(w \cdot \overline{l})$	$(\overline{w} \cdot \overline{l})$
heavy and short	light and short

The student at the Formal Operational Stage can recognize these possible combinations before experimentation and can use them in generating propositions. The pre-formal level student becomes aware of the possible combinations only by manipulating the materials. There are sixteen binary propositions that can be derived from the above four combinations of variables. The propositions are the possible outcomes of the experiment. They are derived by taking each possibility alone, then by taking the combinations of two possibilities, then taking combinations of three possibilities, and finally by taking all four possibilities together (Inhelder and Piaget 1958).

THE SIXTEEN BINARY PROPOSITIONS

Pendulum Experiment	Four possible combinations of two variables in an experiment							
weight and length variables	1		2		3		4	
	$w \cdot l$		$w \cdot \bar{l}$		$\bar{w} \cdot l$		$\bar{w} \cdot \bar{l}$	
Type of Proposition	All ways in which outcomes of the combinations of two variables can be observed							
1. Conjunction	$w \cdot l$		–		–		–	
2. Non-implication ($w \not\leftarrow l$)	–		$w \cdot \bar{l}$		–		–	
3. Non-implication ($l \not\leftarrow w$)	–		–		$\bar{w} \cdot l$		–	
4. Conjunctive Negation	–		–		–		$\bar{w} \cdot \bar{l}$	
5. Independence of w to l	$w \cdot l$	v	$w \cdot \bar{l}$		–		–	
6. Independence of l to w	$w \cdot l$	v	–		$\bar{w} \cdot l$		–	
7. Equivalence (w = l)	$w \cdot l$	v	–		–		$\bar{w} \cdot \bar{l}$	
8. Reciprocal Exclusion	–		$w \cdot \bar{l}$	v	$\bar{w} \cdot l$		–	
9. Inverse of Independence of l to w	–		$w \cdot \bar{l}$	v	–		$\bar{w} \cdot \bar{l}$	
10. Inverse of Independence of w to l	–		–		$\bar{w} \cdot l$	v	$\bar{w} \cdot \bar{l}$	
11. Disjunction	$w \cdot l$	v	$w \cdot \bar{l}$	v	$\bar{w} \cdot l$		–	
12. Converse Implication ($l \rightarrow w$)	$w \cdot l$	v	$w \cdot \bar{l}$	v	–		$\bar{w} \cdot \bar{l}$	
13. Implication ($w \rightarrow l$)	$w \cdot l$	v	–		$\bar{w} \cdot l$	v	$\bar{w} \cdot \bar{l}$	
14. Incompatibility	–		$w \cdot \bar{l}$	v	$\bar{w} \cdot l$	v	$\bar{w} \cdot \bar{l}$	
15. Tautology	$w \cdot l$	v	$w \cdot \bar{l}$	v	$\bar{w} \cdot l$	v	$\bar{w} \cdot \bar{l}$	
16. Negation	$w \cdot l$	v	$w \cdot \bar{l}$	v	$\bar{w} \cdot l$	v	$\bar{w} \cdot \bar{l}$	

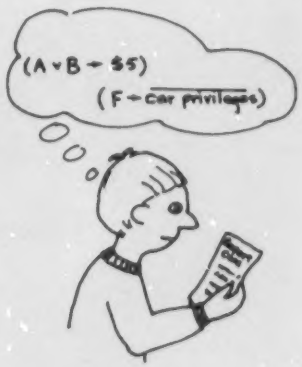
Although the propositions in the example might never be made explicit by the students at the Formal Operational Stage, it is clear that as they carry out experiments, such propositions guide their experimentation. Generally, students will begin with an hypothesis, devise several experiments to test the hypothesis, experiment systematically holding all the variables constant except one, so that the importance of each variable can be evaluated, use information gained to frame new hypotheses, and ultimately draw a conclusion based on all the experimental data.

b. The Specific Model

Theoretically, there are four cognitive operations that Formal Operational Stage students can use when dealing with information. In describing these operations, the following notation is used:

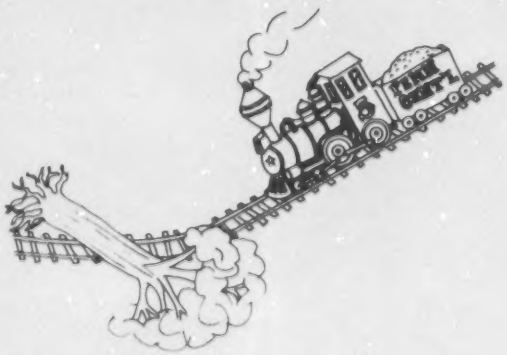
$\rightarrow$ = implies; $\wedge$ = and; v = or; $\bar{}$ = not.

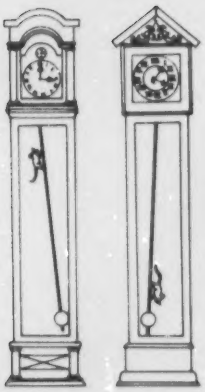
The following are some examples of what Piaget means.



1. *Identity Operation.* This operation is a null operation that changes nothing. The identity operation of "*if x, then y*" is "*if x, then y.*" Thus, I (x→y) = (x→y).

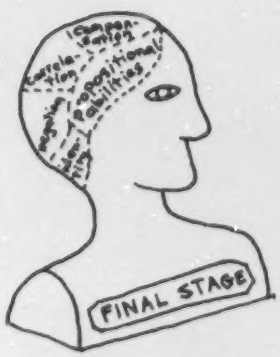
2. *Negation (Inversion) Operation.* This operation negates all aspects of a proposition. Suppose a student has already discovered a series of facts about a problem involving inertia—that a freely rolling train stops because of friction, air resistance, or other variables. This, translated into logical notation is b → f v r v n or "*If the train stops, then there is friction, or air resistance, or other variables operating.*" The negation of this proposition is $\bar{f} \wedge \bar{r} \wedge \bar{n} \rightarrow \bar{b}$ or "*If there is no friction, and no air resistance, nor any other variables operating, then there will be no stopping of the train.*" This states the principle of inertia, which cannot be observed in any concrete form but which can be derived by negating the initial proposition. Thus, through the negation operation all conjunctive propositions can become disjunctive propositions and all assertions can become negations.





3. *Compensation (Reciprocity) Operation.* This operation generates the reciprocal of a proposition. Suppose the variables in the initial proposition were weight and length, then the proposition $w \cdot l$ (heavy and long) can be transformed to the reciprocal $\bar{w} \cdot \bar{l}$ (light and short). The reciprocal operation does not change conjunctive or disjunctive propositions.

4. *Correlation Operation.* The correlation operation alters conjunctive and disjunctive propositions. In a problem about the flexibility of a series of rods, the following disjunctive propositions might be drawn: "*If bending occurs, then either length or the material or both are causing the bending,*" or $b \rightarrow l \vee m$. The correlate of this proposition would be $b \rightarrow l \hat{}$ m.



These two models (The General Model and The Specific Model) form the core of Piaget's theory about the nature of thought at the Formal Operational Stage of cognitive development. The final equilibrated state is based on a completely integrated structure of propositional abilities combined with a completely integrated set of the four operations.

Log

Researchers have carried out fewer studies on the propositional abilities and operations related to the Formal Operational Stage of development than they have on the abilities and operations related to other stages described by Piaget. Research dealing with thinking has increased tremendously in the past few decades; however, the efforts have been primarily directed toward the preschool and elementary levels. Although there is a large number of works on the affective and social life of the adolescent, there is little available on the adolescent's thinking.

a. *Replication Studies*

Carpenter (1955), Lovell and Ogilvie (1960), and Elkind (1961) were among the earliest researchers to replicate systematically Piaget's work on Formal Operational Stage thinking. They found evidence that supported the sequencing of stages and substages proposed by Piaget.

In England, Lovell (1961) repeated ten of the fifteen tasks originally described by Inhelder and Piaget (1958). Two hundred subjects, ages eight to 32, were individually interviewed. The ten tasks used were: flexibility of rods, oscillation of the pendulum, falling bodies on an inclined plane, effects of invisible magnetization, combinations of colorless chemical substances, conservation of motion on a horizontal plane, equilibrium in the hydraulic press, equilibrium in the balance, projection of shadows, and correlations. Although the English subjects found the tasks more difficult than Piaget's Swiss subjects found them, Lovell's conclusions confirmed the sequence that Piaget observed in the development of propositional abilities. Lovell also found that many college students did not perform at the Formal Operational Stage.

Noelting (1961) conducted a longitudinal study using the same students and tasks used by Inhelder and Piaget. His findings corroborated the earlier findings. In addition, he theorized that the structures of thought and those of mathematics were homologous.

The works of Hotyat (1961) and Lee (1968) also confirmed Piaget's sequential developmental pattern. Huttenlocher (1964) compared the thinking of adolescents with that of younger students and found, as Piaget did, qualitative differences between them.

Jackson (1965) investigated students of average IQ (90-110) on a variety of propositional ability tasks: oscillation of the pendulum, falling bodies on an inclined plane, conservation of motion on a horizontal plane, equilibrium in the balance, communicating vessels, and the law of floating bodies. His students ranged from five to 15 years of age. Half of the students ($n=24$) at age 15 were found to be still at the pre-formal level of development. Although the other half were found to be at the Formal Operational Stage, only five of the students had reached the most equilibrated substage (IIIb).

Lovell (1968) conducted a study of gifted students in England, to see if they achieved the Formal Operational Stage earlier than students of average IQ. His subjects, all with IQs above 140, ranged in age from eight to 11.5 years. The three tasks used were oscillation of the pendulum, equilibrium in the balance, and combinations of colorless chemical substances. On these tasks, 15 of the 140 students were found to be at the Formal Operational Stage. Of these, 11 were only at the earliest substage (IIIa). Lovell concluded that gifted students do not acquire formal level abilities until at least age 12.

Keasey (1970) used five of the tasks with females who were college age and above. The five tasks selected were flexibility of rods, oscillation of the pendulum, falling bodies on an inclined plane, combinations of colorless chemical substances, and eqilibrium in the balance. She found that attainment of the highest substage of formal operations was rare. College coeds and older women showed some structural characteristics of propositional abilities, but these characteristics were not stable. The subjects in this study found the tasks more difficult to perform than did the subjects observed by Inhelder and Piaget.

McKinnon and Renner (1971) presented five tasks to college freshmen: flexibility of rods, oscillation of the pendulum, conservation of quantity, angles of incidence and reflection, and the relationship of weight and volume of floating and sinking objects. The researchers found that only the conservation of quantity task was successfully performed by more than 50% of the freshmen.

Lengell and Buell (1972) used the pendulum task and developed a hierarchy of levels that students follow as they solve the problem (right). The researchers claimed that the steps paralleled Piaget's developmental stages. Although the Identification of Factors level corresponded to the earliest stage, it was believed that the Quantification level was more advanced than the IIIb substage. In the study, students moved progressively along the hierarchy from grade seven to grade 12 (80% reached substage IIIb by grade 12).

HIERARCHY

Quantification
↑
Verification
↑
Tentative Conclusions with Exclusion
↑
Identification of Independent Variables
↑
Identification of Factors

b. *Other Studies*

Several investigators have designed their own tasks to examine the Formal Operational Stage of development.

Morf (1957) developed a series of brief stories, each posing a problem that required advanced reasoning for its solution. His procedures included sets of questions and the review of all details of a story until it was completely understood by the student. The most difficult stories were solved by only a few students at 15 years of age.

Case and Collinson (1962) asked students questions concerning selected passages from history, geography, and literature. They then categorized the responses as to the use of intuitive, concrete, or formal operational thought. They reported that "*between 13 and 15 years of age there is a considerable increase in incidence of formal thought, with a corresponding decrease in concrete and intuitive level answers.*" (page 107)

Yudin and Kates (1963) investigated adolescents' attainment of concepts. They found that at age 14 - 16, there were fewer perceptual errors and significantly more consistent and integrated strategies than at age 12. The findings were interpreted as being consistent with Piaget's assertion that formal thought begins at about age 12 and that the propositional abilities are generally well-developed by ages 14 - 16.

Using a sample of students whose IQs ranged from 80 to 130, Yudin (1966) reaffirmed his previous conclusions. He reasserted the constancy of Piaget's developmental sequence, but pointed out variations in the rate at which the stages develop. For example, students in the lowest IQ range did not begin to employ propositional abilities until the ages of 14 - 16, but they did show significant gains during these two years.

Lunzer (1965) analyzed Inhelder and Piaget's original tasks and classified them into two sets. One set required consideration of one variable while holding all others constant; the other set required an understanding of proportionality. He then constructed other tasks to check on Inhelder and Piaget's conclusions. Among the tasks were verbal and numerical analogies. For example, a multiple choice statement required the student to finish the phrase "*leather is to shoe as wool is to...*" with the word *cardigan.* To Lunzer, such an analogy depends upon the abstraction of several relationships (*Leather* and *wool* might be united in the class of *raw materials*; *shoe* and *cardigan* might be in the class of *wearing apparel*; and the finding of second order relationships or conceptual intersections between the first two concepts and the last two. After testing 153 English students between nine and 17 years of age, he concluded that both verbal and numerical analogies require the application of formal reasoning and that formal reasoning is qualitatively distinct from concrete reasoning, because it clearly requires second-level relationships.

Gyr and Fleisher (1967) devised an apparatus which helped them examine college students' abilities to solve a problem. Their data clearly distinguished levels similar to Piaget's stages. They found that 84% of the sample preferred to solve the problem at a concrete, pre-formal level and only 14% solved it using Formal Operational Level thinking.

In his work on adolescent thinking, Peel (1960) found that even capable students were likely to use concrete and less abstract methods because they required less cognitive effort.

Kessen (1960) indicated that, although the student may have the capacity to use propositional abilities, he is not compelled to do so. He may revert to any of the earlier levels of abilities, for earlier levels are not eradicated, but instead are integrated into later stages.

Wason (1968) devised a task which involved the following conditional statement: "*If there is a D on one side of any card, then there is a three (3) on the other.*" When represented in logical terms $(d \rightarrow t)$, the following statements are possible: $(d \cdot t)$, $(\bar{d} \cdot t)$, $(d \cdot \bar{t})$, $(\bar{d} \cdot \bar{t})$. Only the third possibility is logically false given that $(d \rightarrow t)$. The subjects, all university students, were then asked to prove whether the statement was true or false for a set of cards provided by the experimenter. Since this task involves a statement of the form "*if d, then t,*" the solution requires selection of variables and combinatorial reasoning. Wason found that the students performed far below the level expected. Students generally were not able to isolate variables and subject them to combinatorial reasoning. Wason further raised the question: "*Could it then be that the state of formal operations is not completely achieved at adolescence, even among intelligent individuals?*"

Stone and Ausubel (1969) attempted to determine if the transition from the concrete to the formal stage occurs simultaneously in all content areas. To find out, they asked students to read three difficult learning passages and to answer multiple choice questions. Tenth grade students performed at significantly higher levels than did seventh-grade students with comparable IQs. The researchers concluded that formal thought in any subject is not possible until sufficient concrete background experience has been attained in each content area.

Bart (1970) presented a set of tasks to be administered to groups of adolescents. The tasks involved readings and questionnaires covering three subject matter areas: biology, history and literature. He administered the group tasks to students who were 13, 16, and 19 years of age, and supplemented this by administering four of the Piagetian formal operational tasks to each student individually. The correlation coefficients between the Piagetian tasks and each of the subject area questionnaires were "*moderate*" according to Bart. He described his work as "*somewhat successful*" and stated that no valid group testing tool for the Formal Operational Stage was in existance at that time.

Buell and Bradley (1972) presented high school chemistry students with a graph of solubilities and requested that they interpret the graph and tell how they reached their conclusions. The researchers applied the 16 binary propositions in analyzing responses and found 4% of the students to be at the concrete stage and 14% at the formal stage. The remainder were in transition. Following two weeks of instruction on solubilities, the researchers found little change in the percentages. Students could perform less complex binary operations (e.g., conjunction, reciprocity), but not the more complex ones (e.g., implication, tautology).

Lynch (1973) developed a technique for determining levels of large groups of individuals. The group-administered technique obtained more information in a shorter period of time than the usual individually-administered tasks. Lynch found that the technique, a film and questionnaire, was equivalent to Piaget's pendulum task.

Karplus and various collaborators (1970, 1970, 1972, 1973) developed and tested tasks dealing with deductive reasoning and proportional reasoning. The deductive reasoning task required a kind of thinking primarily concerned with the logical coherence of a group of propositions—inferences were to be drawn only after having made an hypothesis as regards the truth of a proposition in question, given that another set of propositions (clues) were true. The ability to do this type of reasoning was present in only 10% of a sample of 152 teachers of science. The sample was composed entirely of persons who taught the principles embodied in many of Inhelder and Piaget's formal reasoning tasks. The proportional reasoning task involved direct proportions and revealed that consistent and correct use of this ability is not yet attained by the majority of above-average 15-year-old students. Since varying the task contexts is no problem for formal thinkers, having attained levels of abstraction that allow sensitivity to form regardless of content, it must be concluded from the students' lack of consistency on a variety of tasks that they were not at the Formal Operational Stage. In fact, these researchers have the impression that a stage of formal reasoning, as characterized by Piaget, may not exist. Their longitudinal research indicates that students' responses improve with age, but that there is no evidence for the existence of an invariant sequence of response categories.

c. *Teaching Studies*

Aebli (1951) devised an experiment to test the idea that if teaching imitated the operational nature of thought, learning would take place. Students were taught a sequence of lessons on the perimeter and area of a rectangle by operational or by non-operational methods. Studen s experiencing the operational method explored and experimented with physical materials and progressed to discussion and exposition of ideals. NOTE: Aebli was not promoting simple activity, but activity that was mobile and reversible and that reproduced the operational nature of the concept. It was found that operational teaching produced more learning than non-operational teaching for average and below-average students.

Lovell (1961) attempted to ascertain the effect of instruction on the level of thinking. His results indicated that knowledge gained by instruction prior to the development of the appropriate level of thinking either disappears or, if it remains, does so as rote knowledge. knowledge.

Boyer (1969) achieved success when he used symbols of logic with abstruse logical statements to develop logical thinking in students.

Keasey (1970) found that her training procedure was very effective in changing problem-solving behavior between pre-test and the immediate post-test, but that there were no gains on a delayed post-test.

McKenzie (1972) was successful when he attempted to develop logical thinking in social studies by giving quizzes that required inferential thinking.

Marek and Renner (1972) used some of Inhelder and Piaget's tasks in a statewide investigation of the thinking abilities of 588 seventh- through twelfth-grade students in Oklahoma schools. They found that the capability to think at the formal level improves through the grades. They also found that Oklahoma's current science program in biology, which requires the student to separate variables, see implications, use the principle of exclusion, and reason with the "*if ... then ... therefore ...*" construct, was appropriate for only 27% of the students enrolled in the program. They recommended that teachers must better select the curriculum on the basis of the students' learning characteristics.

Epilogue

Although Piaget's work suggests that the student moves from the Concrete Stage to the Formal Stage in the period from between the ages of 11 and 16, other research suggests that the level may not be reached until college age, and even then it may never become the preferred mode of cognitive operation for some individuals. In spite of the differences in ages at which the stage and substages appear, the sequence or order of appearance of the abilities seems o have been well corroborated.

Research has not shown clearly whether the Formal Stage can be accelerated by instruction. There is a general indication, however, that the selection of what is to be taught should be better matched to the student's level of development.

We need to learn much more about adolescent thinking.

PIAGET'S DEVELOPMENTAL SEQUENCE

LOWERY'S STRAND

Periods of Development	Ages*	Propositional Abilities
I. Sensory-motor Period (birth to about 2 years)	0	
II. Concrete Intelligence Period (about 2-11 years)	2	
a. Preconceptual Stage (2-4)	4	
b. Intuitive Thinking Stage (4-7)	6	
	8	
c. Concrete Operations Stage (7-11)	10	
III. Formal Operations Period (about 11-18 years)	12	Probabilistic Reasoning (12-13)
	14	Disjunctive Reasoning (13-14)
	16	Combinatorial/Implicational Reasoning (14-15)
	18	Controlling/Manipulating Variables (14-16)
		Deductive Reasoning (15-17)
		Propositional Reasoning (16+)

*All ages listed on this page are approximations

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APPENDIX

Motivation, interest, and persistence often contribute to the attainment of solutions to propositional problems such as those given in Workshop F.

The following are examples of various types of problems related to propositional abilities. If you like thinking through possibilities or combining deduction with analysis, and if you enjoy tasks which require persistence, you and your students might try the following problems.

Although no solutions or analyses are provided for the problems, some references to solutions are given.

Probabilistic Reasoning Problems

Flipping Coins

Find the probability of three pennies falling all heads when tossed simultaneously.

Rolling Dice

Find the probability of a five turning up in one toss of a pair of dice.

Combinatorial Reasoning Problems

Choosing Socks

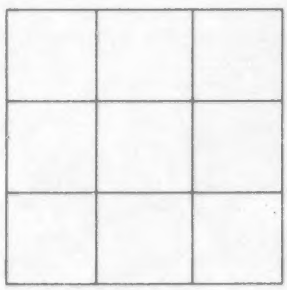
A drawer contains two red socks, two green socks, and two blue socks. What is the least number of socks you can take from the drawer, with your eyes closed, and be sure you have a pair that matches?

Making Change[1]

In how many different ways can a dollar be changed with an unlimited supply of halves, quarters, dimes, nickels, and pennies?

Making a Magic Square[2]

How can nine digits (1, 2, 3, 4, 5, 6, 7, 8, 9) be placed in a square array to form eight intersecting sets of three digits (rows, columns, and main diagonals), each adding up to the same number?



Printing License Plates

How many different license plates can be made if a letter of the alphabet is followed by a three-digit number on each?

Propositional Reasoning Problems

Traveling in the Mountains[3]

In the mountains there is a path where it is forbidden to pass when it is night and when you do not have a lantern. You can pass during the night when you have a lantern or without a lantern when it is daytime. May each of the following people pass or not pass along the path: 1) A tourist who arrives at night with a lantern? 2) A person who comes during the night and does not have a lantern? 3) Someone who has a lantern but it is daytime? 4) A tourist who comes during the day and has no lantern?

Identifying The Beads[4]

A king possessed a coffer filled with large and small beads of many different colors. In the interior of the coffer he placed a small golden box which contained the most precious of his beads and which were also of many different colors and sizes. One day he let his friend look at his beads and pick them up. Since he trusted his friend, he left him alone. But something unfortunate occurred. The friend let the golden box fall into the coffer, and the most precious beads fell among the others. It became no longer possible to distinguish them. The friend called the king's treasurer, because he was afraid to tell the king what had happened. He asked the treasurer for counsel. But the treasurer did not want to explain how the beads had been arranged and only said to him, "*The pink beads which were in the box were not little. There were also some small ones, but they were not pink.*" The friend tried to manage with that information. He first took a large pink bead and asked himself if it could have fallen from the box. He did the same with a small white bead, then a small pink bead, and finally a bead which was neither large nor pink. What did the friend decide for each of these beads?

Controlling and Manipulating Variables Problems

Identifying Combinations of Variables[5]

A hamster is timed to see how fast he can find his way through a maze At the start of the experiment, the first thing the animal must do is pass through either a blue door or a green door. When it has done this, it must pass through a door marked A, a door marked B, or a door marked C. Finally, it must go through either a door numbered 4 or one numbered 5. Once the hamster passes through any door, it cannot go back again. Identify all the different three-door paths that the hamster can take to get through the maze. (Remember that each path has a colored door, a lettered door, and a numbered door.)

Controlling Variables[6]

An agricultural researcher is planning to test three poisons labeled 404, 606, and 707 on two insect pests that are identified as type A and type B. If he can use each poison by itself and mix them together in any way, write the experiments that must be made to test every poison and poison combination on each pest.

Deductive Reasoning Problems

The Trainmen[7]

Smith, Jones, and Robinson are the engineer, brakeman, and fireman on a train, but not necessarily in that order. Riding the train are three passengers with the same three surnames, to be identified in the following premises by a "Mr." before their names.

1. Mr. Robinson lives in Los Angeles.
2. The brakeman lives in Omaha.
3. Mr. Jones long ago forgot all the algebra he learned in high school.
4. Smith beat the fireman at billiards.
5. The passenger whose name is the same as the brakeman's lives in Chicago.
6. The brakeman and one of the passengers, a distinguished mathematical physicist, attend the same church.

Who is the engineer?

The Homeowners[8]

Use the following facts to solve this problem.

1. There are five houses, each of a different color and inhabited by men of different nationalities, with different pets, drinks, and cigarettes.
2. The Englishman lives in the red house.
3. The Spaniard owns the dog.
4. Coffee is drunk in the green house.
5. The Ukrainian drinks tea.
6. The green house is immediately to the rignt (your right) of the ivory house.
7. The Old Gold smoker owns snails.
8. Kools are smoked in the yellow house.
9. Milk is drunk in the middle house.
10. The Norwegian lives in the first house on the left.
11. The man who smokes Chesterfields lives in the house where the horse is kept.
12. Kools are smoked in the house next to the house where the horse is kept.
13. The Lucky Strike smoker drinks orange juice.
14. The Japanese smokes Parliaments.
15. The Norwegian lives next to the blue house.

Now, who drinks water? And who owns the zebra?

The P[illegible]ong Contest

In a ping pong contest, Ken beat Rich and Bill; Rich beat Beth, Nancy, and Bill; Nancy beat Ken and Beth; Beth beat Ken and Bill; and Bill beat Nancy. Rank the players according to their winning ability.

Who's With Whom?[9]

A couple of times a year, in the province of Outer Suburbia, there's a get-together of five congenial families - the ALLISONS, the BAKERS, the CONNORS, the DAVIDSONS, and the ELLENDERS.

Naturally, all the adults are on a first-name basis. The wives are FRANCINE, GLORIA, HELEN, ISABEL, and JUDITH. Their husbands (not respectively!) are KENNETH, LEON, MICHAEL, NORMAN, and OSCAR.

As it happens, each of the adults is presently married for the second time; and what's more, each has been divorced from a member of the group. Each marriage produced one child, and each of the ten adults is the parent of one boy and one girl. The children whose parents are divorced live with their mothers. The get-togethers are attended by all ten children - PETER, QUENTIN, RUTHIE, SANDRA, TED, URSULA, VINCENT, WILMA, XAVIER, and YETTA.

Here are a few facts about the most recent bash:

1. Each couple arrived separately, accompanied by their child and the wife's child by her previous marriage.
   a. Helen and Xavier were among the occupants of the first car.
   b. Ursula and Vincent arrived in the second car.
   c. Leon and Sandra were among those in the third car.
   d. Mrs. Allison and Ted were in the fourth car.
   e. The party was held in the house where Norman and Wilma reside.
2. In the ladies' table tennis competition, the doubles match was won by Michael's wife and Xavier's mother, who defeated the team of Mrs. Connors and Judith.
   a. The two members of the winning team then opposed each other in a singles match in which Yetta's mother defeated Francine.
   b. The members of the losing doubles team also played singles against each other, and Kenneth's wife defeated Sandra's mother.
   c. The winners of the singles matches then opposed each other for the championship. After some brilliant volleying, the former Mrs. Baker finally edged out Gloria.
3. Later there was some really great singing by a quartet consisting of Mr. Davidson, Gloria's husband, Peter's father and Gloria's former husband.
   a. After their second number, Mr. Davidson and one other member of the group decided to take a break, so the remaining two members did a couple of selections with their daughters. Everyone agreed that Ursula, Michael, Ruthie, and Mr. Ellender were pretty good.
   b. Then the two members of the original quartet who had taken a breather returned to the spotlight and did a number with their sons. When they finished, a round of applause went to Kenneth, Quentin, Vincent, and Helen's husband.

c. Finally, the four youngsters who had sung with their dads did some rock 'n' roll improvisations, and the biggest hand of the day went to Quentin, Michael's daughter, Kenneth's son and Ruthie.

4. The only ones who didn't enjoy the day to the fullest were the Bakers.

   a. Mr. Baker had such an acute case of laryngitis that he had to borrow a pen from his son Xavier in order to communicate by written note.

   b. Mrs. Baker had sprained her ankle the day before, and even while leaning heavily on her son Quentin's arm, she was clearly in agony.

On the basis of these facts, you are now ready to figure out:

(1) the full names of the five husbands

(2) who's presently married to whom

(3) whom they had previously been married to

(4) which child is the product of each marriage.

Proportional Reasoning Problems

Finding Values

If y varies directly as x and is 18 when $x = 6$, find the value of y if $x = 2$.

Painting Fence Posts

The amount of paint needed to paint the sides of a cylindrically shaped fence post varies jointly as the radius and height of the post. Compare the amount of paint needed for a post five feet high of radius one-half foot with that needed for a post six feet high of radius one-third foot.

Taking Photographs

The exposure time necessary to obtain a good photographic negative varies directly as the square of the f number of the camera lens. If an exposure of 1/50 second produces a good negative at $f16$, what exposure time would be necessary at a lens setting of $f8$?

Bending Bars

The amount that a bar will bend under a given force varies inversely as the width of the bar when the thickness and length of the bar remain the same. If a bar 2.5 inches wide will bend seven degrees under a certain force, what is the width of a bar of the same material, length, and thickness that will bend five degrees under the same force?

Measuring Distances

One of Kepler's laws states that the square of the time in days required by a planet to make one revolution about the sun varies directly as the cube of the average distance of the planet from the sun. If Mars is one and one-half times as far from the sun, on the average, as the earth is, find the approximate number of days required for it to make a revolution about the sun.

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